On the Relation of the Methods

OF

Just Perceptible Differences and Constant Stimuli

By

SAMUEL W. FERNBERGER

A thesis presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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I. INTRODUCTION

Psychophysics has evolved a number of procedures by means of which the sensitivity of a subject can be determined. These procedures differ widely as to the experimental arrangement which they require and as to the calculation to which the results are subjected, but they have the common purpose of measuring the sensitivity of the subject. We may say, in general, that a psychophysical method is a prescription for the collecting of data and their evaluation in such a way that the result enables us to compare the sensitivity of different subjects, or of the same subject under different conditions or at different times. All these methods agree in this one point, that by them we undertake to give measures of sensitivity. The quantities which are used as the measures of sensitivity are widely different and, perhaps, not always directly comparable. One is confronted with the situation that the comparison of the sensitivity of different subjects by a given method yields perfectly satisfactory results, but that these results do not always agree with those obtained by other methods. One is then led to ask, what the relation of these different measures of sensitivity may be. This problem is frequently stated in the form of the question of the relation of the method of constant stimuli and the method of just perceptible differences.

It has been frequently pointed out that the methods of constant stimuli and just perceptible differences show variations of an experimental or of a mathematical nature in consequence of which their results are not comparable. Numerous investigators have noted this fact and have made different classifications of the methods on the basis of differences in their experimental procedure or in the treatment of the results. In all of these classifications the method of just perceptible differences is always the representative of one group and the method of constant stimuli is given as an example of another group [cf. Titchener, *Experimental Psych.*, II, II, pp. 315-318]. At present these two methods have been developed to a more or less standard form, evidenced by the fact that different authors describe them almost uniformly. The method of just perceptible differences is described by Wundt [*Phys. Psych.,* 5th Ed., vol. I, pp. 470]; G. E. Müller [*Die Gesichtspunkte und die Tatsachen der Psychophysischen Methodik,* pp. 179] and Titchener [*Experimental Psychology,* vol. II, part I, pp. 55-69; part II, pp. 99-143]. A description of the method of constant stimuli is to be found in G. E. Müller [*Gesichtspunkte,* pp. 35] and in Titchener [*Exp. Psych.,* II, part I, pp. 92-118; part II, pp. 248-318].

The solution of the problem of the relation of the psychophysical methods certainly would have been much simpler, had it been possible to subject the same experimental data to the different calculations. The results of experiments made according to these descriptions cannot be treated as material for both methods. It was therefore necessary to divide the experiments into two groups, in one of which the data was taken by the method of just perceptible differences and in the other by the method of constant stimuli; and this brought with it the difficulty of deciding whether a given difference between the results was due to chance variations; to changes in the attitude of the subject, or to differences in the methods themselves. The differences in the arrangement of the experiments by the two methods, indeed, are so great that one may suspect the existence of differences in the attitude of the subject.

It has been argued that the influence of expectation, found in the method of just perceptible differences, where the subject has a knowledge of the stimuli, is a serious handicap. This influence is obviously absent in the method of constant stimuli since the experiments are arranged in such a way that the subject is given no clue whatsoever, as to the objective relation of the stimuli.

In any study of the relation of these two methods, an effort must be made to eliminate all such influences and to have the data in such form that any variations between the results will be due to differences in the methods themselves and not to any of the influences due to the experimental procedure. One means of eliminating the influence of expectation is to mingle the experiments by the two methods in such a way that the subject has no information as to the stimuli compared. The conditions in the two groups of experiments will be very nearly identical, if the results are taken simultaneously, but these conditions may undergo certain changes in the course of the experimentation. This belief cannot be disposed of offhand since the collection of the data necessarily requires a considerable time. One of the most important of these conditions is the psychophysical make-up of the subject. It is very likely that the psychophysical make-up itself does not remain constant since at least one factor, namely practice, changes. The difficulty, therefore, is twofold and requires an investigation of the changes due to variations that have an experimental basis, and also an investigation of the purely formal character of the methods. These two sides of the problem must not be confused. It may well be that the two methods are formally identical but in actual experimentation do not give the same results, since they are performed under different conditions. The suspicion that the conditions are not the same for the methods of just perceptible differences and constant stimuli is very strong, since the entire experimental arrangement is such as to produce different attitudes on the part of the subject. Thus it cannot be expected that the two methods will give the same results unless the experiment is arranged in such a way as to make the conditions directly comparable. Many experimenters have noted the fact that if results were taken by one of these methods and compared with the results of the other, that they do not agree. We may mention Meinong [*Ueber die Bedeutung dest Weberschen Gesetzes. Zeits. f. Psych. und Phys. der Sinnesorgane,* XI, 1896, pp. 244]; Ebbinghaus [*Psych.,* I, pp. 504]; Merkel [*Phil. Studien,* IV, p. 543] and Boas [*Pflüger's Arch.,* XXVIII, 1882, pp. 562].

The investigation of the formal character of the methods is the problem which Urban set for himself, and he devised for this purpose the notion of the probability of a judgment, which logically led to the notion of the psychometric functions. [*The Application of Statistical Methods to the Problems of Psycho-Physics,* 1908, pp. 106.] Forming a judgment on the comparison of two stimuli is a chance event and there exists a certain probability with which a certain judgment will occur under certain conditions. If our experimental data are extended enough, the observed relative frequency of that judgment may be regarded as an empirical determination of its unknown probability. One of the conditions of a judgment is obviously the intensity of the stimuli. The psychometric functions give the probabilities of the different judgments as functions of the intensities of the comparison stimulus. Urban showed that the method of just perceptible differences could be analysed by means of these notions and that its final results could be stated in terms of the probabilities of the different judgments and the intensities of the stimuli. Empirical determinations of the probabilities of the extreme judgments, i. e. of the judgment greater and smaller, are the basis for the calculation by the method of constant stimuli, and it is therefore possible to test the formal character of the methods under discussion on one and the same set of results. Experience showed that both methods of treatment gave essentially the same results.

We are, therefore, confronted with the fact that the methods of constant stimuli and just perceptible differences give the same results if the calculations are made on the same material; but different results if the materials are different. From this we conclude that the methods are formally identical but that the conditions, under which the experimental data must be gained, are materially different; that is, that the methods favor different attitudes of the subject. One must, therefore, devise an experimental procedure by which the two methods can be performed under as nearly identical conditions as possible, in order to study the agreement of the results from the two methods.

The collection of a large amount of material enables one to study incidentally, an entirely different problem; that of whether the conditions remain approximately the same throughout the experiment, or if they undergo certain changes. If the physical conditions of an experiment are kept as constant as possible, so that no variations in the conditions can be attributed to them, we must attribute any varying conditions to a change in the psychophysical make-up of the subject. One factor, at least, making such a change extremely probable is the one due to the practice acquired by the subject during the performance of the experiments.

Extended material, therefore, enables one to study the influence of progressive practice. If the experiments are made on two subjects, one of whom has a high degree of training in this kind of experimentation, while the other has none, one has the opportunity to study the influence of practice in a very advanced stage and to compare it with the practice in the initial stage.

These factors have led to the selection of the experimental arrangement for this study. It enables us to investigate the problems of the formal and experimental relations of the methods of just perceptible differences and constant stimuli, and incidentally, the effect of progressive practice.

My thanks are due to Prof. F. M. Urban for suggesting this problem to me and for acting as a subject. I also have to thank Prof. E. B. Twitmyer for revision of manuscript.

II. ARRANGEMENT OF EXPERIMENTS

This paper is based on the results of experiments in lifted weights on two subjects. The experimentation began January 3, 1912, and was completed February 20, 1912. During this time records were taken almost every day and many times during the morning and afternoon. Subject I, Dr. F. M. Urban, was highly trained in the technique of lifted weights and was the same as the one designated as subject II in a former study by Urban [Statistical Methods]. Subject II, the writer, had some experience in psychological experimentation but, at the beginning of this experiment, had no training in judging small differences of lifted weights. He was given one day's practice, in order to become acquainted with the experimental procedure and with the sensations produced by weights that differ but slightly in intensity. From the second day his judgments were recorded.

The weights used in this experiment were hollow brass cylinders, closed at one end. They were approximately 2.5 inches in diameter and 1 inch high and the wall was 0.0625 inches thick; and they were brought to any desired weight by filling them with shot and parafine. Although these weights had the same outward appearance to the subject, each weight could be recognized by the experimenter by means of small numbers stamped on them with a steel die. The set consisted of 15 weights; of which 7 were standard weights of 100 grams each. The comparison weights for the set used with the method of constant stimuli were 84, 88, 92, 96, 104 and 108 grams. There were also two weights of 84 grams and 97.44 grams which served for the preparation of the variable stimulus in the method of just perceptible differences.

These cylinders had been made slightly lighter than the weight desired and a very delicate adjustment could be obtained by inserting shot and parafine until they were heavier than the proper intensity. The parafine was then carefully scraped out until the desired weight was obtained. An effort was made to use as little parafine as possible since it is susceptible to greater variation from

Number of Weight	Desired Weight	1/2/12	1/8/12	1/15/12	1/22/12	1/29/12	2/5/12	2/12/12	2/22/12
1	100.000	+10	0	+ 5	0	0	— 5	—10	—10
2	100.000	+ 5	+ 5	+10	+ 5	+ 5	+ 5	+ 5	0
3	100.000	+ 5	0	0	— 5	0	—10	—10	—10
4	100.000	— 5	—10	—10	—10	—10	—15*	0	— 5
5	100.000	0	— 5	— 5	— 5	— 5	—10	—10	—10
6	100.000	0	— 5	0	— 5	— 5	—10	—10	—15
7	100.000	0	— 5	— 5	—10	—10	—15*	— 5	— 5
9	97.440	0	0	— 5	—10	—10	—15*	0	0
52	84.000	— 5	— 5	—10	—15*	0	0	— 5	—10
Blank	88.000	0	— 5	— 5	—10	—10	—10	—15*	0
12	92.000	+ 5	— 5	—10	—15*	0	— 5	—10	—10
13	96.000	0	—10	—10	—15*	0	0	0	0
14	104.000	—10	—15*	0	— 5	0	0	—10	—10
15	108.000	— 5	—10	—10	—15*	0	—10	—10	—10
19	84.000	+ 5	0	0	0	0	— 5	— 5	—10

Table I

atmospheric changes than the shot. It was not deemed necessary to determine the weights within a smaller value that 5 mgr. The weights were tested daily for the first week and then once a week throughout the experiment. Whenever a weight was found to vary more than 10 mgr., it was readjusted, but this was necessary only nine times during the experiment. During the first three weeks only one adjustment was necessary and no single weight had to be adjusted more than once. Table I contains the weights of the cylinders and also the amount of variation discovered in them, these variations being given in + or — mgr. from the correct weight. The first column contains the numbers stamped on the cylinders; the next column gives the correct weight of each, and the succeeding columns contain the variations of the cylinders found on the date at the head of the column. A star indicates that a cylinder was found to vary more than 10 mgr. and that it was readjusted.

In the choice of the materials which compose the weight, there are three essential points to be considered. In the first place, the weight must show very little variation from atmospheric changes; secondly, there must be no distinguishing marks on them by means of which the subject can tell one from another. And thirdly, they must be the same in temperature and must arouse the same tactile sensations. Various types of weights have been suggested in the past. The weights used by Fullerton and Cattell were wooden boxes weighted to the proper intensity with shot and raw cotton. They were 6 cms. in diameter and 3 cms. high [Fullerton and Cattell, *The Perception of Small Differences,* 1892, pp. 118]. Jastrow [*Amer. Journal of Psych.,* V, p. 245] used weights of the same type. Galton [*Inquiries into Human Faculty,* 1883, p. 373] used weights that were made by placing shot in a cartridge shell. Urban [*Stat. Meth.,* p. 1] used the same brass cylinders that were employed in the present study. Brown [*The Judgment of Difference, Univ. of California Pub.,* V, I, No. 1, 1910] used cylindrical tin boxes 2.5 cms. high and 4.5 cms. in diameter, which were weighted with shot and parafine.

In the choice of weights for our experiment, the wooden boxes must be at once thrown out of consideration, since they do not fulfil the first of our requirements. The variations in these weights are quite considerable as was shown by Urban [*Stat. Meth.*, p. 173]. In this study Urban gives a table showing the variations in his set of weights, which were similar to those used in this experiment and a set of Cattell weights that were adjusted but not used. The cartridge weights are open to the same criticism, although probably not to as high a degree as the wooden weights. An effort should be made to have the weights consist of as anhygroscopic materials as possible. With the weights used in this experiment, the brass cylinders themselves and the shot are practically anhygroscopic. The parafine was included to keep the shot stationary and to simplify the adjustment and readjustment of the weights.

The weights were as nearly alike as it was possible to turn them out, making it impossible for the subject to distinguish between them. The table that follows contains the height and diameter of each weight to 0.001 of an inch, and it can be seen that these

Weight	Height in inches	Diameter in inches
1	0.996	2.466
2	0.998	2.468
3	0.998	2.470
4	0.995	2.470
5	0.997	2.468
6	0.997	2.464
7	0.998	2.464
9	0.990	2.468
12	0.997	2.468
13	0.999	2.469
14	0.998	2.467
15	0.997	2.468
19	0.996	2.467
52	0.997	2.466
Blank	0.996	2.468

differences can be disregarded. Numbers were stamped on them so that they could be identified by the experimenter. The surfaces were polished and lacquered, rendering them similar to the touch. The weights were kept under exactly the same conditions and furthermore, they were handled the same number of times, so that there was no perceptible difference in temperature. Thus the cylinders used in this experiment seem to conform to all the requirements of a set of weights and furthermore, have the advantage of being easily adjustable.

The weights were placed at regular intervals around the circumference of a circular table with a revolving top, which was 75.5 cms. in diameter and was raised 68.0 cms. from the floor. The top was covered with a layer of prepared cork, which deadened the sound of the weights when replaced on the table. The position of each weight was indicated by a small number on the table. The standard and comparison weights were placed alternately—the standard weights at the odd, and the comparison weights at the even numbers on the table.

The subject was seated in a comfortable position with his right arm supported by a table in such a way that the hand, from the wrist down, hung over the edge. An effort was made to have the edge of the supporting table strike approximately the same position of the forearm of the subject. The turn top table was then brought into such a position that, with merely a downward movement of the wrist, the hand would grasp one of the weights.

The cylinders were lifted with the right hand; most of the weight being sustained by the thumb, second and third fingers, and the first and little fingers resting on the edge. The movements of the hand were regulated by the beats of a metronome, which was adjusted to 92 beats per minute, while every fourth beat was accentuated by the automatic stroke of a bell. These hand movements were regulated in the following manner. At the start of each trial, the hand of the subject was raised at the wrist, with the forearm remaining on the table. At the stroke of the bell, the hand was dropped and the weight, which had been brought directly underneath by turning the table, was grasped. At the second stroke of the metronome the weight was lifted, and at the next stroke, it was replaced on the table. Finally at the fourth stroke, the empty hand was lifted, returning to its original position. Between the third stroke of the metronome and the bell following, the experimenter turned the table so that the next weight to be lifted, was directly under the hand of the subject, and everything was ready for the next lifting. In a very short time these wrist movements became quite automatic. The weights were lifted from 2 to 4 cms. and an effort was made to have the height of lifting constant for each subject. Due to the control of the metronome, each weight was in the air approximately the same length of time.

A screen was placed between the subject and the table so that it was impossible for the subject to see the weights; his hand passing through a slit in this screen. Furthermore both subjects voluntarily closed their eyes while making judgments, as they believed that they could make their judgments with more accuracy in that way.

Previous investigators have not made a very clear analysis of the space error. They have sought to avoid it or eliminate its influence rather than explain it. The obvious method of eliminating the space error and the one that has been most frequently used, is to perform the experiment twice, in both of the spatial relations; that is, with the standard weight to the left and right of the comparison weight. Then the error of the one spatial relation counterbalances the error of the other, which is in the opposite direction. In the present study the space error was avoided in a simpler manner; since by means of the revolving top of the table, all side movements of the hand were eliminated. The table was turned so that the weight to be lifted was brought directly under the hand of the subject. Thus the only movements necessary were in one direction merely—directly downward. Care was taken that the subject did not reach to one side or the other in grasping a weight, since in this case a space error would have occurred. An effort has been made before to avoid the space error by experimental technique rather than eliminate it by repeating the experiment in the two spatial relation. [L. Steffens, *Zeitschrift f. Psych. und Physiol. der Sinnesorgane,* XXIII, 1900, pp. 279, and J. Fröbes, *Zeit. f. Psych. und Physiol. der Sinnesorgane,* XXXVI, 1904, pp. 234]. In these studies the weights were placed on a board which was pushed along under the hand of the subject, so that the weights in turn were directly underneath. This procedure, although it eliminated the space error, had the disadvantage that when a series had been taken, the board had to be replaced in its original position before a second series could be begun. With our experimental arrangement, however, any number of series could be run off in succession. Besides this, the table, which revolves very easily, can be moved with greater regularity and accuracy than could ever hoped to be obtained with a sliding board.

The time error was present in our experiments and no effort was made to either avoid or eliminate it. The standard stimulus was always lifted first and the comparison stimulus second. The metronome controlled all hand movements and kept them regular so that the time error was constant throughout the experiment. As the investigation of the sensitivity of the subjects was not of primary interest, and as a constant time error should not effect the relationship of the results obtained by the two methods under discussion, no attempt was made to avoid this error.

The comparison weights were placed about the table in a carefully arranged order. This order was changed four times during the experiment, partly to eliminate the influence of the particular arrangements, partly to counteract the influence of the knowledge about the arrangement used, which the subjects might have acquired. Table II gives these orders. The first column

Table No.	1/3/12	1/10/12	1/22/12	2/18/12
1 and 2	96	84	84	88
3 and 4	104	104	104	C
5 and 6	108	96	C	104
7 and 8	84	C	88	96
9 and 10	92	92	92	84
11 and 12	C	108	108	92
13 and 14	88	88	96	108

TABLE II

gives the table numbers in pairs: 1 being the first standard stimulus, 2 the first comparison stimulus, 3 the second standard stimulus and so forth. The other columns give the comparison weights at the even numbers of that pair. The C indicates the comparison weight of the method of just perceptible differences series. Each column is under the date at which the order was adopted. In every order, no matter how carefully planned, there are certain landmarks by means of which the subject can tell in what part of the series he is. Such a landmark is seen in the first series where the two heaviest weights (104 and 108 gms.) come together. Another occurs in the second order where the two lightest comparison weights [84 and 88 gms.] came together. Both subjects acted as experimenters and so necessarily became acquainted with the order in which the weights were presented. Furthermore, the experiments were conducted daily and so the subjects became acquainted with the orders more rapidly than if there had been a longer interval between experimentation.

One complete revolution of the table involved the passing of seven judgments: six on invariable comparison weights for the method of constant stimuli, and one on a variable comparison weight for the method of just perceptible differences.

This seventh comparison weight, indicated by C in table II, was adjusted for the method of just perceptible differences in the following manner. At each revolution of the table, a judgment was passed between it and a standard weight in the same manner as the other pairs. After a judgment had been taken on each of the seven pairs, steel bearings of a given number were placed in cylinder C and another complete revolution of the table was made. Then the same number of bearings were placed in cylinder C. This was continued until C weighed over 108 gms., the weight of the heaviest stimulus used in the method of constant stimuli. The bearings which weighed 0.42 gms., were of a surprising uniformity in weight. All the bearings were weighed and they did not show any variations within 5 mgr., which was the limit of exactitude in our weighing. We at first intended to use shot for the purpose of weighting our variable stimulus, but found that the differences among them were quite considerable. Cotton wool was placed in the bottom of the cylinder so that the total weight was 84.80 gms. The cotton wool was placed in the cylinder to keep the bearings from moving about and thus by the noise, indicating to the subject which cylinder he was lifting. The bearings made a noise only three or four times during the lifting, and each time the judgment was thrown out and another taken.

Ten series were taken by means of this method. In series II the variation was two bearings for each revolution of the table. For series III the variation was three bearings, and so on until series X, when a variation of ten bearings was used. In an effort to test the "carefully graded approach" of the central intensities of the comparison stimulus, which is considered by some psychologists to be the keynote of the method of just perceptible differences, another series was planned [Titchener, *Experimental Psychology,* II, II, p. 103]. In this series the two extreme variations were seven bearings; the next two variations toward the central values were six bearings; then in order five, four, three, two, and the five central values varied only one bearing. This is designated as series I. It was not deemed profitable to perform a series with the variation of only one bearing, as this would have necessitated 54 revolutions of the table to complete each series.

Ten determinations of each series were taken; of these five were ascending and five descending. In the ascending series the proper number of bearings were placed in the cylinder, after each revolution of the table; while in the descending series, the total number of bearings were placed in the cylinder at the start of the experiment and after each revolution of the table, the proper number were removed. It is obvious that these series varied in length; the series X required only seven revolutions of the table while series II required 29 revolutions. As a matter of technique, two weights were used for this comparison stimulus. The first with the cotton wool weighed 84.80 gms. and the second with the cotton wool, 98.24 gms. In an ascending series, the 84.80 gm. cylinder was first judged empty, then successive judgments were taken until it weighed equal to or just heavier than 98.24 gms. Then the other cylinder was substituted and the lifting continued. The opposite procedure was used in a descending series. The use of only one cylinder would have necessitated the handling of twice as many bearings as were used.

The series were not taken in any regular order but entirely at haphazard; the determining factor in the choosing of a series being the amount of time at our disposal. If we had sufficient time, a long series of short steps was chosen; while if our time was short, a short series of long steps was used. A different starting point was chosen for each successive revolution of the table. This was done so that, even though a subject knew an order fairly well, he would not be able to tell at what part of that order he had started. It was furthermore deemed advisable, on each revolution of the table, to allow the subject to make two judgments that were not recorded. This was done so that the movements of the wrist might become as automatic as possible and also to give the subject an opportunity to concentrate his attention.

At the beginning of experimentation each day, the subjects made one complete revolution of the table grasping the weights as strongly as possible and lifting them high and vigorously. This was done for a double purpose: it assured the experimenter that the weights were in the correct position on the table in relation to the hand of the subject. At the beginning of experimentation, the weights give the impression of great lightness which is lost as the lifting process proceeds. This process can be hastened by the sort of lifting just described, and after such a warming up, both subjects made introspections that they had little trouble with the absolute impression.

After each five to seven revolutions of the table, the subject was allowed to rest, until he was willing to resume experimentation. If a subject declared that he felt fatigued or unfit he was not asked to experiment.

In the manner just described, results by the method of just perceptible differences were taken simultaneously, and therefore under as nearly indentical conditions as possible, with results by means of the method of constant stimuli. The former are the results obtained with the weight C and a standard weight, and the latter are the judgments on the six other pairs of weights.

Immediately after each comparison weight had been replaced on the table, a judgment was given in terms of the comparison weight. These judgments were given verbally and by saying one word, three terms being used: heavier, equal and lighter. A heavier judgment signified that the comparison weight was subjectively heavier than the standard weight just preceding it.

A lighter judgment signified that the comparison weight was subjectively lighter than the standard weight just preceding it. The equality judgment was more complex as it not only included cases of actual subjective equality between the standard and comparison weights, but also all those cases where it was impossible for the subject to give either a lighter or a heavier judgment, usually termed doubtful cases. The cases of absolute subjective equality were much more frequent with subject I than with subject II.

The results were recorded by the experimenter on printed blanks of the following form:

96	h	e	e
104	h	h	h
108	h	h	h
84	l	l	l
92	l	e	l
C	l 0	l 6	l 12
88	l	l	l

The first column of these blanks gives the comparison weights, C being the variable weight for the method of just perceptible differences. The succeeding columns give the judgments for a complete revolution of the table, H signifying a heavier; E an equal, and L a lighter judgment. The small numbers (0, 6, 12, etc.) indicate the number of bearings in the comparison weight C during that revolution of the table. The above chart is a portion of the record of a series VI, as the weight C is regularly varied by six bearings.

From Fechner down, there has been a great deal of discussion about the choice of judgments; Fechner himself, objecting to the use of the equality or doubtful judgments [*Elemente der Psychophysik*, I, 72, 94. *Revision der Hauptpunkte der Psychophysik*, 67]. He suggested that the equal and doubtful judgments be divided in half and that one half be added to the right and the other half to the wrong cases. This procedure was followed by

Fullerton and Cattell [*Perception of Small Differences,* pp. 59]. Merkel suggested that the equal judgments be not only excluded from the calculations but also from the records [*Philosophische Studien,* IV, 131] and this method was followed out by Kraepelin in his experimental procedure [*Philosophische Studien,* VI, pp. 496]. Jastrow [*Amer. Jour. of Psych.,* I, 282] and Higier [*Phil. Stud.,* VII, 247] contended that there should be no equal or doubtful judgments and that when the subject does not know whether the judgment should be greater or less, he should guess. Sanford [*Course,* pp. 357] follows this same procedure. Brown [*Judgment of Difference,* 1910] is the latest adherent of the exclusion of the equality or doubtful judgments. He even asserts that if the subject is forced to give a judgment of either greater or less, that he can do so. Brown apparently fails to notice that this places his uncertain judgments in exactly the same category as those of Jastrow where the subject is forced to guess. Nor should the numerical results of these variations differ widely from those of Fechner, because the laws of chance would give approximately an equal division of this class of judgments between the two other classes. The only difference is that Fechner makes this division frankly while the others hide it under the technicalities of experimental procedure.

Urban [*Application of Statistical Methods,* pp. 5] uses a very complicated system of judgments. He uses the classes of greater, equal and lighter. The degree of confidence with which the judgment is given is designated by the subject by the numerals 1, 2 or 3. Besides these he uses two guess classes, "Guess-heavier and guess-lighter". Later in the paper [*Stat. Meth.,* pp. 14] he states that this system is too complicated and that the guess judgments are not advisable as they afford a loophole for the subject who does not wish to commit himself. Among the men who favor the use of the equality judgment we find Ebbinghaus, Wundt and Müller.

Titchener [*Experimental Psychology,* vol. II, part II, pp. 290] points out the original reasons why the equal judgments were first excluded. He shows that it was because three classes presented difficulties in mathematical treatment that were at that time unsurmountable. But these difficulties have since been solved and a further exclusion of judgments of this class for that reason, he calls unscientific. Titchener further points out that we have the evidence of introspection of trustworthy observers that equal and doubtful judgments do actually occur. "If we are dealing with mind as mind presents itself to us for examination, we cannot ignore these judgments." This seems to be an unanswerable criticism. Brown, in his paper, gives a case of subjective equality. His subject stated [*Judgment of Difference,* pp. 30] "those two are exactly the same". This may have been only one case in 4000, as Brown states, but still it must be accounted for. The subjects in our experiment both reported actual subjective equality; subject I gave this introspection quite frequently. The criticism of Brown [pp. 28] that the classes should be mutually exclusive is just as applicable to three classes as to two. His criticism that [pp. 32], for the equality judgment, the subject must maintain a mental standard of equality is not very impressive. Undoubtedly the subject must maintain such a standard, but he must also maintain, in the same sense, a standard for heaviness and lightness.

The classes of judgments chosen for this experiment fulfil all the requirements of the case. They are susceptible to mathematical treatment and they are not so complicated that the subject has any difficulty in giving them. The subject is not put to the strain of forcing a guess judgment. Lastly and of primary importance they fit the facts of actual experience.

III. METHOD OF CONSTANT STIMULI

The record sheets enable us to find the relative frequencies of the heavier, equal and lighter judgments for all of the six comparison weights. In order to study the effect of practice, these observed frequencies were divided into groups of 100 in the order in which they were taken. These results are given in Table III for subject I and in Table IV for subject II. The numbers in the first column give the groups of 100 judgments in the order in which they were taken. Three columns are given to each comparison weight in which appear the lighter, equal and heavier frequencies of the judgments on that weight. The numbers in the same columns show, on the whole, a rather close agreement, but that they are by no means identical. For this reason it is not possible to say offhand whether a constant effect of practice has taken place or whether the conditions remain the same. This decision may be affected in two ways; by the determination of the coefficient of divergence for the probabilities for the different judgments, and by the determination of the constants for the psychometric functions.

The data in tables III and IV are results of repeated observations of certain unknown probabilities and the question arises whether the conditions, which determine these probabilities, remain the same or undergo variations. The coefficient of divergence [Lexis and Dormoy] enables us to make this decision systematically. If the coefficient gives a value close to unity we have a normal dispersion, and may know that the conditions have been approximately constant during the experiment. If the coefficient of divergence is considerably smaller than unity we obtain an under normal dispersion and we conclude that there is a law or rule which tends to produce always the same value of the relative frequencies. If the value comes out considerably greater than one, we speak of a more than normal or over normal dispersion which indicates that the conditions during the experiment have varied. We may illustrate these three kinds of dispersion by

	84			88			92			96			104			108		
Hundreds	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier
1	0.94	0.05	0.01	0.83	0.16	0.01	0.55	0.32	0.13	0.35	0.44	0.21	0.06	0.21	0.73	0.00	0.08	0.92
2	0.94	0.06	0.00	0.78	0.21	0.01	0.69	0.21	0.10	0.32	0.40	0.28	0.06	0.18	0.76	0.02	0.07	0.91
3	0.96	0.04	0.00	0.90	0.08	0.02	0.60	0.23	0.17	0.37	0.41	0.22	0.02	0.15	0.83	0.03	0.06	0.91
4	0.93	0.07	0.00	0.93	0.07	0.00	0.80	0.13	0.07	0.33	0.42	0.25	0.07	0.16	0.77	0.03	0.06	0.91
5	0.97	0.03	0.00	0.81	0.18	0.01	0.61	0.32	0.07	0.37	0.34	0.29	0.02	0.16	0.82	0.02	0.04	0.94
6	0.98	0.01	0.01	0.80	0.16	0.04	0.57	0.26	0.17	0.39	0.42	0.19	0.09	0.16	0.75	0.02	0.04	0.94
7	0.95	0.05	0.00	0.91	0.05	0.04	0.60	0.25	0.15	0.46	0.37	0.17	0.09	0.17	0.74	0.01	0.04	0.95
8	0.98	0.02	0.00	0.89	0.10	0.01	0.60	0.23	0.17	0.45	0.32	0.23	0.06	0.13	0.81	0.03	0.06	0.91
9	0.95	0.04	0.01	0.88	0.11	0.01	0.69	0.25	0.06	0.42	0.31	0.27	0.07	0.12	0.81	0.00	0.02	0.98
10	0.99	0.01	0.00	0.92	0.06	0.02	0.76	0.17	0.07	0.47	0.33	0.20	0.10	0.11	0.79	0.00	0.06	0.94
11	0.96	0.03	0.01	0.83	0.15	0.02	0.56	0.36	0.08	0.41	0.36	0.23	0.11	0.14	0.75	0.03	0.04	0.93
12	1.00	0.00	0.00	0.90	0.09	0.01	0.63	0.23	0.14	0.50	0.35	0.15	0.05	0.16	0.79	0.01	0.09	0.90
13	1.00	0.00	0.00	0.94	0.06	0.00	0.72	0.21	0.07	0.49	0.41	0.10	0.07	0.18	0.75	0.00	0.03	0.97
14	0.97	0.02	0.01	0.91	0.08	0.01	0.63	0.26	0.11	0.45	0.40	0.15	0.03	0.14	0.83	0.02	0.12	0.86

TABLE III

	84			88			92			96			104			108		
Hundreds	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier
1	0.85	0.09	0.06	0.70	0.20	0.10	0.47	0.26	0.27	0.39	0.19	0.42	0.14	0.08	0.78	0.03	0.08	0.89
2	0.94	0.03	0.03	0.77	0.17	0.06	0.60	0.24	0.16	0.49	0.27	0.24	0.10	0.12	0.78	0.04	0.08	0.88
3	0.97	0.02	0.01	0.91	0.04	0.05	0.64	0.19	0.17	0.47	0.22	0.31	0.05	0.10	0.85	0.04	0.04	0.92
4	0.99	0.00	0.01	0.90	0.09	0.01	0.63	0.23	0.14	0.42	0.26	0.32	0.01	0.09	0.90	0.01	0.03	0.96
5	0.96	0.04	0.00	0.88	0.07	0.05	0.54	0.26	0.20	0.47	0.26	0.27	0.00	0.08	0.92	0.01	0.03	0.96
6	1.00	0.00	0.00	0.85	0.13	0.02	0.58	0.28	0.14	0.37	0.31	0.32	0.01	0.03	0.96	0.00	0.04	0.96
7	0.95	0.05	0.00	0.79	0.16	0.05	0.59	0.26	0.15	0.38	0.33	0.29	0.03	0.12	0.85	0.03	0.06	0.91
8	0.95	0.04	0.01	0.79	0.17	0.04	0.56	0.29	0.15	0.26	0.36	0.38	0.02	0.13	0.85	0.01	0.09	0.90
9	0.94	0.06	0.00	0.82	0.14	0.04	0.55	0.33	0.12	0.25	0.37	0.38	0.03	0.14	0.83	0.01	0.04	0.95
10	0.98	0.02	0.00	0.84	0.14	0.02	0.61	0.19	0.20	0.28	0.37	0.35	0.07	0.15	0.78	0.02	0.07	0.91
11	1.00	0.00	0.00	0.75	0.23	0.02	0.52	0.30	0.18	0.31	0.35	0.34	0.05	0.14	0.81	0.00	0.03	0.97
12	0.97	0.03	0.00	0.89	0.09	0.02	0.69	0.23	0.08	0.32	0.47	0.21	0.08	0.19	0.73	0.02	0.09	0.89
13	0.96	0.03	0.01	0.90	0.08	0.02	0.68	0.23	0.09	0.30	0.40	0.30	0.05	0.33	0.62	0.00	0.03	0.97
14	0.98	0.02	0.00	0.88	0.12	0.00	0.66	0.28	0.06	0.46	0.33	0.21	0.03	0.15	0.82	0.01	0.07	0.92

TABLE IV

considering the drawing from an urn of black and white balls of a given relative frequency. Several urns are prepared and n numbers of drawings are made from each urn, and each ball is replaced after its colour has been noted. Under such conditions if the number of drawings is great enough, we will obtain a coefficient of divergence approximating unity, or in other words, a normal dispersion. An over normal dispersion will be obtained when the urns, before the drawings are made, contain different relative frequencies of black and white balls. Again each ball will be replaced in the urn after its colour has been noted and the same number of drawings will be made from each urn. But the varying frequencies in the urns represent changed conditions so that our results will give a coefficient of divergence greater than unity. If we can by some kind of device eliminate certain cases we will obtain an undernormal dispersion. For example, if we decide that every time three white or black balls are drawn in succession we shall record the third ball as having the opposite colour, we would obtain a coefficient of divergence less than unity. From this we conclude that in an under normal dispersion the results have been tampered with in some way. We must calculate the coefficient of divergence for the probabilities of each judgment for every intensity of the comparison stimulus that we used. The formula by which the coefficient of divergence is caluclated is

$$Q = \sqrt{\frac{s \Sigma v^2}{(n-1)\ p\,(1-p)}}$$

in which s is the number of observations in each series; Σv^2 is the sum of the deviations of the relative frequencies from their average, n is the number of series and p the average of the relative frequencies, which is the most probable determination of the probability of the judgment. The quantity (1 — p) will then be the probability that this judgment will not occur. Table V is a double table that gives the coefficients of divergence for both subjects I and II. In the first column are found the intensities of the comparison weights. The next three columns give the coefficients of divergence for the heavier, lighter and equal judgments for subject I. In the vertical column, headed Average, is given the average of the coefficients of all three judgments for

	Subject I				Subject II			
Stimulus	Lighter Judgments	Equal Judgments	Heavier Judgments	Average	Lighter Judgments	Equal Judgments	Heavier Judgments	Average
84	1.236	1.223	0.935	1.131	0.582	1.387	1.720	1.230
88	1.531	1.568	0.971	1.357	1.690	1.517	1.243	1.483
92	1.544	1.346	1.306	1.399	1.238	0.883	1.464	1.195
96	1.157	0.842	1.292	1.097	1.672	1.551	1.313	1.512
104	1.100	0.699	0.815	0.871	1.752	1.970	2.108	1.943
108	1.004	1.087	1.124	1.072	1.116	0.996	1.216	1.109
Average	1.262	1.128	1.074	1.154	1.342	1.384	1.511	1.412

TABLE V

each comparison weight; and in the horizontal column we find the averages of the vertical columns. The last value of the horizontal averages is the total average of all the coefficients of divergence. The second half of the table shows a similar distribution for subject II.

In the case of subject I, the total average, 1.154, is well within the limits of what may be considered a normal dispersion, which indicates that the conditions remained constant for this subject. Such individual variations as the coefficient for the equal judgments of the 88 gm. weight [1.568] or that for the equal judgments of the 104 weight [0.699] may be accounted for by the chance variations of the results. The values of the coefficient of divergence are dependent, to a certain extent, upon the size of the sum of the deviations. It will be noted that all of the averages for subject I are outside of the limits of what must be considered an over normal dispersion.

For subject II, on the other hand, the total average is 1.412, which indicates a somewhat overnormal dispersion. All of the averages but one, the average for the 92 weight, are higher than the corresponding values for subject I. Four of the averages for subject II are above the limit for the over normal dispersion and two others are just below it, so that they almost certainly indicate the same tendency. This result indicates that a change in the conditions which determine the probabilities of the different judgments has taken place and we will have to consider what this change may have been.

The passing of a certain judgment may be dependent upon either the physical conditions of the experiment or upon the psychophysical make up of the subject, or to a complex of both influences. The physical conditions of the experiment remained as constant as it was possible to control them, hence the variations must have had their origin within the subject himself. We may liken the psychophysical make up of an individual to our example of the black and white balls in an urn, in which those psychophysical influences that control the passing of a certain judgment are likened to the balls. Now if these influences remain constant we will obtain a normal dispersion as in the first example where the relative frequencies of the white and black balls in the urns were the same. The normal dispersion for subject I would indicate such a constancy of conditions. The overnormal dispersion in the results of subject II indicates an influence which was at work in this subject but not in subject I, who showed approximately a normal dispersion. The changes in subject II may be likened to the case where the relative frequencies of the black and white balls varied for the different urns, and it seems to be an obvious idea to see this influence in the progressive practice acquired during the experiments. It will be remembered that subject I was very highly trained in the lifting of weights; while subject II had only one day's training at the start of the experiment; just enough to enable his hand movements to become somewhat automatic. It is, therefore, very likely that the changes in the conditions, as indicated by the coefficients of divergence, were due to the effect of practice. Similar results were found by Urban [Psych. Massmeth., *Arch. f. ges. Psych.,* XV, p. 283], where it was found that the subjects with the most considerable training showed very nearly a normal dispersion.

The changed conditions, as indicated by the coefficient of divergence greater than unity, may be due to a complex of four influences: first, a change in the sensitivity; second, to an unconscious learning of the order of the stimuli; third, to fatigue; and fourth, to the effect of practice. Practice itself is a complex of two elements that are, however, closely related and dependent upon one another; first, the acquiring of the automatic movements of the hand and by this the direct effect of the elimination of the space error and constancy of the time error. Second, the directing of the entire attention on the judgments, when it is no longer necessary to direct some of it upon the hand movements.

It is possible that there may be a certain training of the sensitivity in the same way that we may have training in a muscle, bringing with it increased efficiency. The sense organs employed in this experiment, the end organs of touch in the hand and the free nerve endings in the wrist and the forearm, are end organs constantly in use. So it is reasonable to believe that they are normally at a high state of efficiency so that this factor of the training of these sense organs cannot be of considerable extent.

The influence of fatigue could not have been great, as during the entire time of the lifting, we endeavored to eliminate this factor by resting the subject after every 5-7 revolutions of the table. Besides, the passing of a judgment on the intensities of two weights where the differences were as small as in this experiment, would be impossible in a state of fatigue, since in that case, the judgment would really become nothing but a guess. But even if we have present some influence of fatigue, this would be practically eliminated, since we are comparing groups of 100 experiments, each of which represent three or four days' experimentation. Thus there would be several places in the series, where the subject was fresh and as many places where he was fatigued, and these influences would tend to cancel one another.

It seems a reasonable expectation that the effect of practice would be greatest in the beginning of the experiments and that the conditions should remain constant after a certain perfection is attained. In order to test this supposition, the coefficient of divergence was calculated for subject II for the last 13 groups, omitting the first group of 100 experiments. The amount of work in this second set of calculations, can be greatly reduced by deriving the new sum of the squares of the deviations from the values already ascertained. This step, which is the one that requires the most time in the calculations, may be accomplished by applying the formula

$$\Sigma\, v^2_1 = \Sigma\, v^2 + nd - (A^1 - a)^2 .$$

In this formula, n represents the number of groups; d the difference between the average of the 14 groups and the average of the 13 groups, which latter is represented by A^1; and finally, a stands for the omitted result.

Table VI, which has the same form as table V, gives the new values of the coefficient of divergence for subject II for the last 13 groups only. It will be noticed that, although the individual values change, the total average for the two sets of calculations for this subject are almost identical. This would indicate that the changed condition, whatever it may be, did not have its greatest effect in the first series. Where an individual coefficient in table VI is smaller than the corresponding value in table V, it

	Subject II			
Stimulus	Lighter Judgments	Equal Judgments	Heavier Judgments	Average
84	1.180	1.223	1.306	1.236
88	2.473	1.551	0.906	1.643
92	1.107	0.953	1.265	1.108
96	1.802	1.440	1.236	1.493
104	1.492	2.051	2.282	1.942
108	1.195	1.047	1.262	1.168
Average	1.542	1.377	1.376	1.432

Table VI

indicates that the deviation in the first series was large. The opposite holds for the cases where the terms in table VI are larger than those in the former calculation. It will be seen, however, by the averages that, on the whole, these variations almost cancel one another. If the over normal dispersion was due to the effect of practice, we may know that this was not as rapid in the first series as might have been expected, but that it was gradual throughout the experiment.

A further treatment of the coefficient of divergence strengthens the belief in the importance of the concentrating of the attention upon the judgments. If the averages for each intensity of the comparison stimulus for both subjects are averaged, we obtain the values

84 — 1.180
88 — 1.415
92 — 1.295
96 — 1.402
104 — 1.329
108 — 1.116

It will be noticed that for the extreme values of the comparison stimulus, the averages of the coefficients of divergence are small and that, with the exception of the average for the 92 weight, they gradually rise toward the central values. Considering the fact that these are averages of the results of only two subjects, it is remarkable that this course should be broken at only one place. The explanation of this set of values seems to lie in the fact that it takes less concentration of the attention to judge the difference between an 84 or a 108 gram weight and a standard weight of 100 grams; than it would between the same standard weight and a comparison stimulus that was but little different from it. So at the start of the experiment, subject II could give enough attention to the judgments on the extreme intensities of the comparison weight to have these judgments fairly accurate. It required, however, so much more attention to give correct judgments on the central values that, at the start of the experiment, these were judged much more inaccurately. Later when more attention could be given to the judgments, the central values could be judged more accurately, while there could be little change in the extreme values, as they had been judged with a fair degree of accuracy from the beginning. Thus there was greater variation for the central values than for the extreme ones, and this fact is shown by the size of the coefficients of divergence for the different intensities of the comparison stimulus.

Although the two elements of the automatic movements of the hand and the increased attention on the judgments are interrelated, still it does not imply that when the automatism is perfect, that we have at once a maximal attention on the judgments. It is our belief that for subject II, the hand movements became automatic very early in the experiment and this led us to calculate the coefficient of divergence for this subject for the last 13 groups of experiments. We found, however, but little change in the value obtained from that corresponding value for all of the experiments. This would indicate that the element of practice of the automatic hand movements is not of very great direct importance. This element, however, is fundamental for that of the proper direction of the attention upon the judgments, as the latter influence implies that the automatisms have reached a degree of perfection.

The coefficient of divergence merely shows that a change in the conditions has taken place. It furthermore, indicates the nature of this change but, by no means, can it be taken as conclusive evidence. The examination of the constants of the psychometric functions of the two subjects will give us a chance to study this variation a little more closely. The psychometric functions for the heavier judgments gives the probability of a heavier judgment as function of the intensity of the comparison stimulus. Similarly the psychometric functions for the lighter judgments gives the probability of a lighter judgment; and for the equal judgments gives the probability of an equal judgment, as function of the intensity of the comparison stimulus. If the psychometric functions with all their constants are given, we are able to calculate the probabilities of the different judgments for every intensity of the comparison stimulus. We may represent the course of the psychometric functions graphically, by constructing the comparison weights on the abcissa and the corresponding probabilities of the different judgments as ordinates. The curve representing the psychometric functions for the lighter judgments will set in with high values [close to unity], and will drop at first slowly, then more rapidly and eventually it will approach the abcissa asymtotically. The psychometric functions for the heavier judgments will have just the opposite course, setting in with low values and attaining the values close to unity for the high intensities of the comparison stimulus. The curve for the psychometric functions of the equality judgments starts with low values, then rises to a maximum and finally falls off very rapidly. A diagram of this kind enables us to see the variations of the probabilities of the different judgments at a glance.

It would be the same if we were to get an analytic expression to express this set of facts. The choice of such a mathematical formula will be in the nature of a hypothesis about the psychometric functions. The one chief requisite of such an expression is that it fits the facts of the observed frequencies. Several such hypotheses can be advanced but they do not fit the facts of lifted weights, for example, as was shown by Urban in regard to the Lagrange formula [Method of Constant Stim., *Psych. Review,* XVII, p. 234]. The $\Phi(\gamma)$ hypothesis recommends itself, however, by its simplicity and the fact that it is known by large experience. This experience has also shown that the $\Phi(\gamma)$ hypothesis comes very near the truth in so far as lifted weights are concerned. This hypothesis underlies the method of constant stimuli, which is essentially nothing else but the determination of the quantities h and c, two values upon which the form and position of the curves of the psychometric functions depend. The greater the value of h, the steeper the curve is and thus the h exerts an influence upon the form of the curve. The influence of c upon the position of the curve is such that a larger c means the shifting of the entire curve to the left. The essential feature of the $\Phi(\gamma)$ hypothesis is that only the values of the extreme judgments are calculated. The values of the equality judgments are found by the difference from unity of the sum of the probabilities of the heavier and lighter judgments. The extreme judgments are those which are greater or less. The intermediate judgments admitted in any study must be odd in number and in this study it was limited to one, the equality judgment. The $\Phi(\gamma)$ hypothesis consists in the supposition that the psychometric functions of the smaller judgments are represented by expressions of the form

$$p = \tfrac{1}{2}\,[1 - \Phi\,(h_1\,x - c_1)],$$

and for the greater judgments by

$$p^1 = \tfrac{1}{2}\,[1 + \Phi\,(h_2\,x - c_2)].$$

In these equations x represents the intensity of the comparison stimulus and Φ has the sign of its argument. If we substitute the relative frequencies of one of the extreme judgments, e. g. of the lighter judgments, for the values of the comparison stimulus used in our experiments, we obtain a series of equations of the form

$$p_{84} = \tfrac{1}{2}\,[1 - \Phi\,(h\,x_{84} - c)]$$
$$p_{88} = \tfrac{1}{2}\,[1 - \Phi\,(h\,x_{88} - c)]$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

from which we have to determine the unknown quantities h and c. It is not possible to solve these equations in this form and so they are changed to read

$$\Phi\,(h\,x_{84} - c) = \frac{2p_{84} - 1}{2}$$

$$\Phi\,(h\,x_{88} - c) = \frac{2p_{88} - 1}{2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

from which the arguments ($hx - c$) may be determined, which is the form to be used in our observation equations. The relative frequencies for all the judgments of all the six comparison stimuli are found in tables III and VI. The values of $\Phi(\gamma)$ corresponding to these relative frequencies are found either in a table of the probability integral or in the so-called fundamental table for the method of right and wrong cases that was calculated by Fechner. It was found that many of the values in Fechner's original table were wrong in the last decimal place because the tables of the probability integral that were in Fechner's possession were not complete. With the appearance of complete tables of the probability integral by Bruns [*Wahrscheinlichkeitsrechnung und Kollektivmasslehre.*, 1906], the fundamental table was recalculated by Urban [Method of constant stimuli, *Psych. Rev.*, XVII, p. 251]. This situation was not understood by Brown [*Mental Measurement,* p. 134, 1911], who prints this table and states that it was calculated by Bruns and quoted by Urban.

These determinations of unknown probabilities, however, are not exactly correct and unless these differences are allowed for, certain discrepancies will appear in the results. To eliminate these errors, each observation equation is given a weight, which is in relation to the probable errors of the observed relative frequencies. G. E. Müller was the first to see that these observation equations are not of the same weight, but did not arrive at the correct formula. Urban took up this analysis and published a table of these weights, calculated by another formula [*Psych. Review,* XVII, p. 253, and *Arch. f. ges. Psych.,* XVI, p. 371. also quoted by Brown, *Mental Measurements,* p. 135].

The values for γ are substituted in the observation equations and it is found to be convenient to write the weight or P after each equation. Thus we obtain an observation equation and a weight for each of the six comparison stimuli used, in this form

$$hx_1 - c = \gamma_1 \text{ with weight } P_1$$
$$hx_2 - c = \gamma_2 \text{ with weight } P_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$hx_6 - c = \gamma_6 \text{ with weight } P_6.$$

This gives an over determined set of equations for the determination of the constants h and c. The solution of this set, by the Method of Least Squares, gives the most probable values of the unknown quantities. For the purposes of calculation an adding machine was found to be invaluable. After some practice, it became possible, with the help of this machine, to effect the solution of such a set of observation equations in one and a quarter hours, which is considerably less than the time required to do the same calculation even with the help of logarithms.[1] The calculations are very easy but rather lengthy so that it is impossible to be sure of their correctness unless the whole work is arranged systematically [cf. Urban, *Arch. f. ges. Psych.*, XVI, pp. 375-377, and Wirth, *Psychophysik*, pp. 210-214]. The scheme used for this purpose is a modification of the Gaussian method for solving a system of equations by the method of least squares. The first step for this solution consists in setting up the normal equations which have the form:

$$[xxP]h - [xP]c = [x\gamma P]$$
$$-[xP]h + [P]c = -[\gamma P].$$

These normal equations require the calculation of the sums of the products xP, xxP, γP, and xγP. The sum of the P is found directly by addition. We obtain the products xP by multiplying each P by its corresponding x; their sum is designated by [xP]. The values xxP are obtained by multiplying each xP by the corresponding x, a multiplication which is performed on the adding machine in two steps without clearing the machine. The calculation of [γP] and [xγP] is similarly arranged. The products γP are formed first and then used for the calculation of xγP without clearing the machine.

These six sums are all the values that are necessary for the setting up of the normal equations for h and c. In order to check the correctness of these results, three other values are calculated. A quantity s is defined as the algebraic sum of all the coefficients of the observation equations, e. g. by

$$s = x - 1 - \gamma$$

[1] Since the writing of this paper, Urban *(Hilfstabellen fur die Konstanzmethode. Arch. fur die ges. Psych. Bd.,* XXIV, 2-3 Heft, pp. 236-243) has printed a set of tables which are of great assistance in this kind of work. By means of these tables a set of equations can be solved in little more than half an hour.

notice being taken of the sign γ. Multiplying this equation by P we obtain

$$sP = xP - P - \gamma P.$$

If these products are formed for every observation equation and added we obtain

$$[sP] = [xP] - [P] - [\gamma P].$$

The terms on the right side of this equation are needed for setting up the normal equations and the calculation of the sum [sP] gives us a thoroughgoing check of our results for the other three sums. As all the decimal places were retained in the terms of this equation it must be expected to solve exactly. Multiplying the equation for s by xP gives

$$xsP = xxP - xP - x\gamma P$$

and by adding up these relations for all of the values of x we obtain

$$[xsP] = [xxP] - [xP] - [x\gamma P].$$

The terms on the right side of the equation are the sums which are used in the normal equations and the sum [xsP] affords a check on our calculations of these sums. Our calculations were arranged in such a way as to require this check to tally to the third decimal place. The check worked out in this way is a great saving of time as it requires only the calculation of three sums [s], [sP] and [xsP]; the s being calculated almost directly. The only other check would be a recalculation of the other five quantities. This would be not only more laborious but furthermore in such a recalculation, it would be quite possible to repeat an error that had been made before.

Tables VII and VIII give these sums for the lighter and heavier judgments respectively for subject I and tables IX and X are similarly constructed for subject II. The first columns give the number of groups of the hundred experiments into which the results were divided. The succeeding columns give in order the values of [P], [xP], [xxP], [γP], [xγP], [sP] and [xsP]. On account of the lack of space these tables are somewhat reduced, by omitting one decimal place for the values [xP], [xxP], [γP], [xγP] and [sP]; and two for the quantity [xsP]. All the

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
1	3.447	319.64	29748.02	— 0.311	—16.982	316.508	29445.37
2	3.619	338.00	31719.38	— 0.182	— 0.728	334.567	31382.10
3	3.215	301.26	28354.91	— 0.211	— 4.999	298.260	28058.64
4	3.256	307.26	29153.89	— 0.291	— 9.421	304.279	28856.04
5	3.307	308.32	28858.56	— 0.222	— 7.379	305.235	28557.62
6	3.611	340.59	32265.26	0.090	23.164	336.887	31901.51
7	3.449	324.32	30629.90	— 0.330	—16.133	321.197	30321.72
8	3.371	319.08	30337.04	— 0.084	7.421	315.796	30010.62
9	3.281	305.82	28610.21	— 0.630	—46.667	303.165	28351.06
10	2.952	279.25	26504.06	— 0.528	—39.168	276.824	26263.98
11	3.806	359.58	34140.69	0.018	18.378	355.756	33762.73
12	2.962	280.37	26621.31	— 0.119	— 1.149	277.525	26342.09
13	2.714	257.32	24461.30	— 0.328	—22.878	254.938	24226.85
14	3.146	295.38	27845.12	— 0.345	—18.502	292.575	27568.24

TABLE VII

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
1	2.975	293.51	29092.88	— 0.465	—30.379	290.998	28829.75
2	2.871	285.30	28459.58	— 0.140	— 0.981	282.565	28165.27
3	2.922	287.94	28497.20	— 0.230	— 8.405	285.248	28217.66
4	2.605	260.96	26228.14	0.050	15.756	258.300	25951.19
5	2.570	254.70	25335.34	— 0.078	4.803	252.208	25075.84
6	3.123	304.70	29878.99	— 0.670	—48.696	302.247	29622.98
7	2.898	284.19	27992.74	— 0.548	—39.625	281.842	27748.17
8	2.901	286.84	28476.46	— 0.146	— 1.135	284.081	28190.76
9	2.430	237.72	23351.30	— 0.443	—30.094	235.728	23143.58
10	2.575	255.01	25357.42	— 0.300	—15.960	252.737	25118.37
11	2.870	282.78	27996.27	— 0.494	—32.552	280.404	27746.04
12	2.753	273.56	27297.22	— 0.232	— 9.431	271.040	27033.08
13	2.068	206.54	20694.72	— 0.207	—10.838	204.683	20499.01
14	2.820	279.67	27878.16	— 0.377	—19.745	277.225	27619.24

TABLE VIII

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
1	4.439	415.04	39024.59	0.244	39.656	410.353	38569.90
2	4.033	379.92	35978.96	— 0.156	2.771	376.039	35596.27
3	3.363	318.46	30300.72	— 0.214	— 3.962	315.307	29985.79
4	2.819	263.38	24677.50	— 0.388	—26.546	260.945	24440.67
5	3.005	277.43	25686.78	— 0.704	—57.119	275.131	25465.98
6	2.727	253.39	23584.27	— 0.222	—15.362	250.883	23346.24
7	3.585	334.70	31390.82	— 0.145	1.862	331.256	31054.26
8	3.289	304.10	28220.03	— 0.117	2.036	300.924	27913.90
9	3.331	308.52	28688.50	— 0.079	6.622	305.264	28373.36
10	3.358	316.41	29944.67	0.138	28.056	312.912	29600.20
11	3.111	290.71	27237.62	0.291	34.752	287.306	26912.34
12	3.312	312.79	29676.58	— 0.098	6.821	309.578	29356.96
13	3.021	281.19	26262.06	— 0.424	—27.844	278.592	26008.72
14	3.079	288.16	27062.00	— 0.454	—30.397	285.531	26804.24

Table IX

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
1	4.160	400.05	38709.09	— 0.511	—28.978	396.399	38338.02
2	3.569	347.26	33990.62	— 0.737	—51.923	344.424	33695.29
3	3.233	314.38	30726.54	— 0.435	—24.809	311.581	30436.84
4	2.635	256.09	24996.58	— 0.303	—15.042	253.756	24755.53
5	2.769	267.35	25921.31	— 0.400	—24.997	264.983	25678.95
6	2.364	228.94	22256.16	— 0.317	—18.810	226.897	22046.02
7	3.092	302.77	29784.58	— 0.280	—11.909	299.956	29493.72
8	3.261	318.42	31249.15	— 0.282	—10.101	315.437	30940.83
9	2.931	286.49	28119.66	— 0.189	— 4.406	283.750	27837.58
10	3.213	316.37	31280.19	— 0.070	6.580	313.225	30957.24
11	2.861	279.23	27351.49	— 0.232	—10.461	276.593	27083.56
12	2.884	287.25	28730.35	— 0.320	—17.755	284.688	28460.85
13	2.925	286.66	28215.86	— 0.730	—58.191	284.466	27987.39
14	2.388	239.19	24038.27	0.059	16.861	236.745	23782.22

TABLE X

checks are fulfilled so that we are sure that no mistake has been made in the calculation of these sums.

These values are substituted in the normal equations which must be solved for the two unknown quantities h and c. It is found to be simplest in most cases to solve for h directly, and then substitute the value found, in the normal equation for c; as in this equation the terms are smaller. A check on this part of the work is effected by substituting the values obtained for h and c in the normal equation for h and observing whether the equation proves. This check will not come out exactly as only four decimal places are retained in the calculations, but we placed the arbitrary limit that the difference between the values of $[x\gamma P]$ obtained should not be greater than one per cent of that value.

After we have obtained the values of h and c, the calculation of the threshold is very simple. The threshold in the direction of increase is defined as that point in the curve of the heavier judgments where the probability of a heavier judgment is ½. Similarly that place in the curve for the lighter judgments where the probability of a lighter judgment is ½, defines the threshold in the direction of decrease. At such a point, according to the $\Phi(\gamma)$ hypothesis the value of $\gamma = 0$. Then

$$0 = hx - c.$$

Then we obtain the formula

$$S = \frac{c}{h}$$

which is used for the calculation of the threshold in the direction of increase if the h and c for the heavier judgments are used. If the constants for the lighter judgments are substituted in the formula, the S will be the threshold in the direction of decrease. Tables XI and XII give the h, c and S values for subjects I and II respectively. Opposite the numbers for the series, given in the first columns, will be found the values of h, c and S for first the lighter and then the heavier judgments. The columns headed S_1 contain the threshold in the direction of decrease and those headed S_2 give the threshold in the direction of increase.

Series	Lighter Judgments			Heavier Judgments		
	h_1	c_1	S_1	h_2	c_2	S_2
1	0.11051	10.337	94.54	0.11371	11.374	100.03
2	0.10769	10.108	93.86	0.11914	11.888	99.78
3	0.11267	10.617	94.23	0.11629	11.537	99.21
4	0.10976	10.442	95.13	0.11734	11.731	99.98
5	0.11714	10.988	93.81	0.13496	13.406	99.33
6	0.10432	9.814	94.08	0.11059	11.004	99.50
7	0.11156	10.586	94.89	0.11393	11.362	99.72
8	0.11379	10.796	94.88	0.11641	11.560	99.30
9	0.11477	10.889	94.88	0.13843	13.724	99.14
10	0.12274	11.790	96.06	0.13364	13.351	99.90
11	0.09902	9.351	94.43	0.12086	12.081	99.95
12	0.12287	11.670	94.99	0.12004	12.012	100.07
13	0.12794	12.251	95.76	0.14643	14.725	100.56
14	0.12446	11.795	94.77	0.12390	12.421	100.25

TABLE XI

Series	Lighter Judgments			Heavier Judgments		
	h_1	c_1	S_1	h_2	c_2	S_2
1	0.07570	7.023	92.77	0.08126	7.936	97.67
2	0.09224	8.728	94.62	0.09763	9.706	99.42
3	0.10859	10.338	95.22	0.10666	10.500	98.45
4	0.13786	13.018	94.42	0.12340	12.108	98.12
5	0.10672	10.087	94.52	0.12580	12.291	97.70
6	0.13459	12.587	93.52	0.14124	13.812	97.80
7	0.10743	10.070	93.74	0.11233	11.090	98.72
8	0.12512	11.604	92.73	0.11112	10.937	98.43
9	0.12402	11.511	92.81	0.12023	11.816	98.28
10	0.11502	10.797	93.87	0.10467	10.328	98.67
11	0.10223	9.460	92.53	0.12377	12.161	98.25
12	0.11750	11.126	94.70	0.11801	11.865	100.54
13	0.13071	12.307	94.16	0.10954	10.985	100.29
14	0.12920	12.239	94.73	0.13650	13.648	99.98

TABLE XII

By a close study of these two tables [XI and XII] we can discover many facts that were only hinted at in the results of the coefficient of divergence. We will consider the effect of the unconscious learning of the order of presentation of the stimuli. The order of the series was changed after the first, fifth and thirteenth groups of 100. If there was any effect of this knowledge of the series, we should expect to find that the h for the group immediately before the change would be larger than the h in the group following. As has been noted above, the h controls the steepness of the curves and with a knowledge of the order we may expect a greater accuracy in the judgments. An examination of the tables will show that for subject I, the h's for the groups immediately after these changes are slightly smaller for the most part, than those of the series just preceding, but this is not true for subject II. This would indicate that for the latter, the influence of the knowledge of the order was of no consequence. For subject I this knowledge may have had some effect upon the results, but as the differences are small this effect was not considerable.

For the convenience of studying the effect of practice, it was deemed advisable to plot the four curves of the course of the h's shown in figures I and II. Upon the abcissa are laid off the series in order; while the ordinates represent the values of h. These curves are strikingly different in form when we compare those of the different subjects; but for the the same subject they are quite similar in character. Although all the curves show unsystematic variations, the curves for subject I start with values that are fairly high and the general trend is upward. This is particularly noticeable in the curve h_2 for subject I. The curves h_1 and h_2 for the other subject start with comparatively small values and, for the first four groups with h_1, and the first six groups for h_2, they show a very rapid rise. Both curves then fall from this maximum, and from the seventh series on, show a very gradual upward tendency.

For reasons stated above we have ruled out the influence of the knowledge of the orders as being negligible. Changes in sensitivity, due to the weather conditions, the tone of the subject and

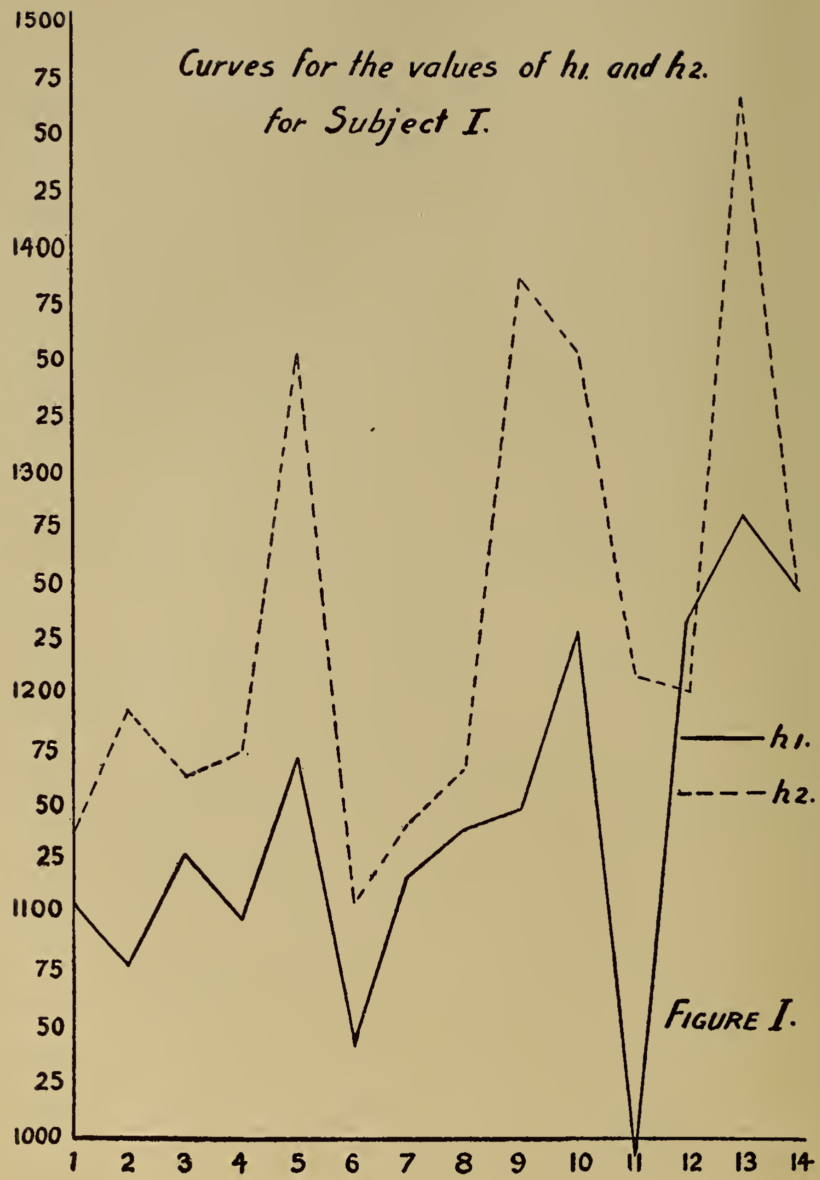
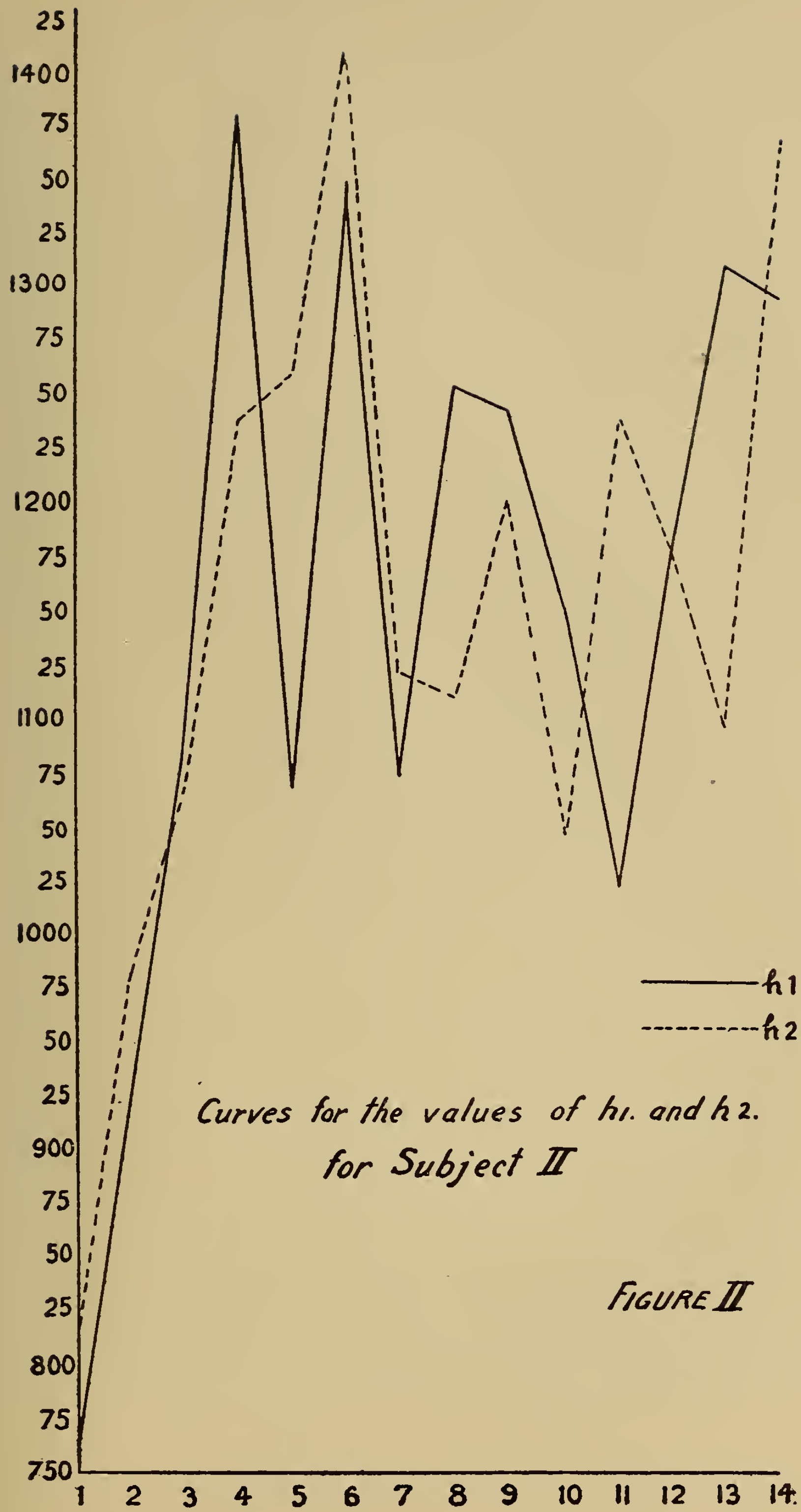

Curves for the values of h1. and h2.
for Subject I.
1500
75
50
25
1400
75
50
25
1300
75
50
25
1200
75
50
25
1100
75
50
25
1000
1 2 3 4 5 6 7 8 9 10 11 12 13 14
h1.
h2.

FIGURE I.



Curves for the values of h1. and h2. for Subject II

FIGURE II

like influences may account for the unsystematic variations or the chance irregularities in the curves. But we have still to account for the systematic tendencies that give the curves for the two subjects their distinctive character. We have eliminated all of the factors that could influence our judgments except that of practice; in fact the curves for subject II might be taken for typical practice curves.

This agrees with the facts regarding the subjects themselves, as it will be remembered that subject II was absolutely untrained at the start of the experiment; while subject I was highly trained, due to his having acted as subject in previous experiments. The values of h start high for the latter, but, in spite of all his former practice, the curve for subject I still shows a gradual rise. The curves for subject II indicate, by their rapid rise in the first six series, that during this time the subject became practised in the method of lifting the weights. This means that the hand movements became so automatic that the space error was entirely eliminated. This may perhaps need some explanation. At the start of the experiment there was a tendency on the part of subject II to turn his hand slightly in the direction from which the weights approached, and thus a space error, even though a slight one, was committed. This, however, was subsequently eliminated when the hand movements of the subject became automatic, and therefore accurate in regard to movement in space. The hand movements improved in regularity as to time and so the time error became more constant as the experiment proceeded. There was still another effect of the perfecting of the automatism, since when perfected it no longer requires any of the attention of the subject, and this then, can be entirely focused on the judgments. In this connection, it will be remembered, that the results of the calculation of the coefficient of divergence argue for the same notion. The drop in the curves after the maximum had been reached cannot be accounted for unless the maximum itself is caused by a chance variation that accentuates it. If a similar curve be constructed for subject I from the results of the former experiment in which he acted as subject, it will show a similar rapid rise in the early series. Several years have elapsed between that experiment and this one. If the sense organs had been trained due to the practice several years ago, to a higher state of efficiency; it is reasonable to believe that due to the disuse during the interval between the experiments, they would have degenerated somewhat to the condition in which they were before they had received any training at all. The rise in the curves for subject II are not very great and this would argue against any considerable education of the sense organs.

It must further be remembered that the measure of the sensitivity of the sense organ is not to be found in the abruptness of the curve of the psychometric functions, h, but by the interval between the two thresholds. This is the basis of the practicability of the method of constant stimuli. An examination of Tables XI and XII will show that the thresholds remain fairly constant. These values show chance variations but it is impossible to discover systematic tendencies in their course. They remain conspicuously constant for subject I. Thus the training of the sense organ, which might directly effect the sensitivity seems to have little influence, if any, as the sensitivity remains very constant and at least shows no systematic variations. This is an important consideration, because if practice effected the measure of sensitivity, the method of constant stimuli would have to be abandoned as a practical means of arriving at that value. It will be noticed, however, that, although the values of the h and c vary considerably, $\frac{c}{h}$ the threshold, remains practically constant.

The characteristics of the curves for the two subjects agree with the values of the coefficient of divergence obtained for the same set of results. For subject I we obtain a coefficient of divergence that approximates unity more closely than that for subject II, and the curves of the former show not only smaller unsystematic variations, but smaller systematic ones as well. Our experiments were probably not extended enough to show long periodicity and the results were not regular enough to reveal short periodicity.

It then seems that the important variations in the curves of the h and c values are:

1. Those of an unsystematic or chance character, that are due to the condition of the subject and similar chance influences.

2. Variations of a systematic nature that seem to have their origin principally as the result of practice which allows a concentration of the attention primarily upon the judgments, when the hand movements become so automatic that they no longer require attention.

3. This effect of practice is not evenly distributed throughout the experiment, but for an untrained subject is comparatively great in the first few hundred experiments.

4. This effect of practice, however, does not seem to effect the real purpose of the experimentation, which is to ascertain the sensitivity of the subject, as the ratios that define the measure of sensitivity remain practically constant and show no systematic variations.

We will turn now from the study of the effect of practice, as shown in this set of results, to a consideration of the results themselves. The real purpose of the psychophysical methods is to ascertain the sensitivity of the subject. The problem is how much larger or how much smaller a comparison weight must be from the standard weight until a difference can be perceived between them. This statement implies that there is a distance on each side of the standard weight, within the limits of which a comparison stimulus, although physically lighter or heavier than the standard weight, will not be judged lighter or heavier with a probability of 0.50. This distance is called the interval of uncertainty and is defined as the difference between the two thresholds. The measure of sensitivity is one half this interval. For example, if we have a standard weight of 100 grams and our interval of uncertainty is 4 grams, the measure of sensitivity will be 2 grams. This means that either a weight of 102 or of 98 grams will be distinguished from a weight of 100 grams with a probability of 0.50.

But these are not all the considerations that are necessary to tell the sensitivity of our subject. It will be remembered that our results are effected by constant errors. Two weights of 100 grams if lifted in either different spacial or temporal relations will not be subjectively equal. In our experiments the space error was eliminated by the turning top table, but the time error was present, although it was controlled by the regular beats of the metronome. We must now find the point of subjective equality, which is that weight which, under the conditions of our experimental procedure, will be subjectively equal to our standard weight. This is found by applying the formula

$$\xi = \frac{c_1 + c_2}{h_1 + h_2}.$$

When this point of subjective equality has been ascertained, we can find the influence of the constant errors almost directly, by finding the difference between the point of subjective equality and the standard stimulus. We will consider how this effects our example given above. Suppose we find that the influence of the time error is — 3 grams. In this case our point of subjective equality is 97 grams. The measure of sensitivity remains the same, however. Thus a restatement of our results indicates that without the influence of the time error, weights of 95 or 99 grams can be distinguished from one of 97 grams.

Table XIII is a double table containing the values for our subjects of these quantities we have just discussed. For each subject will be found in order the values of the interval of uncertainty; the point of subjective equality; and the time error. These will be found opposite the number of each of the groups of 100 experiments into which our results were divided. At the bottom will be found the averages for each of the columns.

It will be noticed that the interval of uncertainty remains very constant for both subjects although the values are by no means identical. The variations are obviously due to changes in the sensitivity of the subject. These are the changes that are caused by the variations in the psychophysical makeup of the individual. There does not seem to be any correlation between the character of these changes for the two subjects. We see that the averages of the interval of uncertainty are almost identical for the two subjects. This is of course a matter of chance and merely indicates that the sensitivity of the two subjects happened to be almost the same.

The time error was a negative one in the Fechnerian sense; that is, the second weight lifted was relatively heavier. This means that the point of subjective equality is smaller than the standard weight. Thus, on the average, for subject I for our arrangement of the experiment, a standard weight of 100 grams would be subjectively equal to a comparison weight of 97.31 grams. It will be noticed that the point of subjective equality is practically midway between the two thresholds. This indicates that the amount that must be added in order to perceive a difference between the two weights is approximately the same as the amount that must be subtracted in order to just perceive a difference. For our calculations the point of subjective equality remained comparatively constant for each subject, but there is considerable difference [one gram on the average] for the two subjects. The time error for subject I seems to become less as the experiment

Number of Groups	Subject I			Subject II		
	Interval of Uncertainty	Point of Subjective Equality	Time Error	Interval of Uncertainty	Point of Subjective Equality	Time Error
I	5.49	96.83	— 3.17	4.90	95.30	— 4.70
II	5.92	96.97	— 3.03	4.80	97.09	— 2.91
III	4.98	96.76	— 3.24	3.23	96.80	— 3.20
IV	4.85	97.64	— 2.36	3.70	96.17	— 3.83
V	5.52	96.76	— 3.24	3.18	96.24	— 3.76
VI	5.42	96.87	— 3.13	4.28	95.72	— 4.28
VII	4.83	97.34	— 2.66	4.98	96.29	— 3.71
VIII	4.42	97.12	— 2.88	5.70	95.42	— 4.58
IX	4.26	97.21	— 2.79	5.47	95.50	— 4.50
X	3.84	98.06	— 1.94	4.80	96.16	— 3.84
XI	5.52	97.47	— 2.53	5.72	95.67	— 4.33
XII	5.08	97.49	— 2.51	5.84	97.62	— 2.38
XIII	4.80	98.32	— 1.68	6.13	96.95	— 3.05
XIV	5.48	97.50	— 2.50	5.25	97.43	— 2.57
Average	5.03	97.31	— 2.69	4.86	96.31	— 3.69

TABLE XIII

proceeds. The physical conditions remained the same throughout the experiment as this constancy is attested by the fact that there are no systematic variations in the course of the time error for subject II. This systematic change, found in the results of subject I is not very great and thus may be due to the chance arrangement of the results. If it is significant, however, it must be due to some change in the psychophysical organism. It is not due to an effect of practice, since on the basis of the discussion of practice given above, we should expect to find an even stronger tendency in subject II, where it is entirely absent.

IV. METHOD OF JUST PERCEPTIBLE DIFFERENCES

The classical form of procedure of the method of just perceptible differences is to start with two stimuli at subjective equality and increase the comparison stimulus C by equal steps only up to the point where a difference is first noticed; this point we call the just perceptible positive difference. We then start again, to obtain the just imperceptible positive difference, with the comparison stimulus C subjectively greater than the standard stimulus S; then C is decreased until no difference is perceived between C and S. Again we start with C and S subjectively equal and decrease C until it is first judged less and thus obtain the just perceptible negative difference. Finally, to obtain the just imperceptible negative difference, we start with C subjectively less than S and increase C to the point where it is first impossible to perceive a difference. Then the just perceptible and imperceptible positive differences are averaged and we obtain the threshold in the direction of increase. The threshold in the direction of decrease is found by averaging the just perceptible and imperceptible negative differences. This is the form in which this method was described by Fechner [*Elemente der Psychophysik,* I, p. 72], Wundt [*Phys. Psych.,* I, 1902, p. 475], Müller [*Gesichtspunkt und die Tatschen der psychophysischen Methodik,* 1904, p. 179], and Titchener [*Exp. Psych.* II, I, p. 56].

Another form of the method of just perceptible differences is the one used in this experiment, which decreases the amount of experimentation considerably and has also other advantages. We start with C so much lighter than S that there is a very small probability of any but a lighter judgment, and then increase the comparison stimulus C by successive steps until there is a very small probability of any but a heavier judgment. The four differences, from which the thresholds are obtained, are then picked out of the results and are defined in the following manner. The just perceptible negative difference is the greatest stimulus upon which a lesser judgment is passed. The just imperceptible negative difference is the smallest stimulus upon which a not-lighter—either equal or heavier—judgment is given. The smallest stimulus upon which a heavier judgment is given becomes the just perceptible positive difference. Finally, the greatest stimulus upon which a not-greater—equal or less—judgment is passed is the just imperceptible positive difference. Thus in one experimental operation we obtain all the values that in the other form of the method required four distinct series. After these differences are obtained, the thresholds are found in the same manner as in the other method. Besides the matter of expediency, there is another great advantage of this form of the method over the classical form. The original method requires that we start at the point of subjective equality; but this term is not defined and the method gives no indication as to what intensity of the comparison stimulus is to be used for it. In actual practice this method is handled in such a way that the comparison stimulus is used, as a starting point, on which an equality judgment is given. This definition, however, is very loose since a great number of comparison stimuli fulfill this requirement. The variation of the method now under consideration, avoids this disadvantage by not making use of the notion of subjective equality at all. The fact that this variation gives satisfactory results indicates clearly that only the so-called upper and lower limits of the interval of uncertainty are of consequence for the determination of the sensitivity of the subject. From this it seems to follow that the discussion as to whether the arithmetic mean or the geometric mean ought to be taken as representative of the point of subjective equality, cannot be decided on the ground of this method; since it does not give any definition of that value whatsoever. [Wundt, *Phys. Psych.,* I, 5th Ed., p. 477.]

In order to avoid the so-called errors of expectation, a descending series of just the reverse experimental procedure was taken for every ascending series, of the form that has been described. This influence of expectation may be twofold; either, if the subject knows that he is in an ascending series, he may give a heavier judgment too soon, or that knowledge may bias him against giving such a judgment. Similarly he may give a lighter judgment too soon or not soon enough, if the subject knows that he is in a descending series. There is, however, a great difference of opinion as to the extent of the influence of expectation. For example, Wundt [*Phys. Psych.,* I, 1902, p. 479, 491] believes that this tendency may be overcome by practice; and Fullerton and Cattell [*Small differences*] think that this influence is so great that they strongly recommend that the subject should have no knowledge of the type of series that he is judging. In many cases it is not possible to keep this knowledge from the subject. For example, in the present study, the subject could tell from the sound made by the bearings being placed in the cylinder, whether he was judging an ascending or descending series. In the old form of the experimental procedure of this method, it was not possible to keep this knowledge from the subject because, after the first judgment, he could realize which type of series he was judging. On the other hand, Titchener [*Exp. Psych.* II, II, p. 128] states that with good subjects this expectation has no influence whatsoever. Whether this is true or not, the method of double procedure, used in this study, allows for this error if present. Since it is not possible to avoid this error, one tried to eliminate it, or at least, to minimize its influence by using ascending and descending series alternately. The errors being in opposite direction in the series may be expected to cancel one another.

We will now analyze the nature of the four differences upon which this method is based. It was believed for a long time, that the method of just perceptible differences uses notions which are foreign to the error methods. Urban, however, has shown that the common source of both methods is to be found in the notion of the probability of a certain judgment. The just perceptible negative difference may be defined as the largest stimulus on which a lighter judgment was given. This implies that none of the greater comparison stimuli were judged lighter, but that on this one a lighter judgment was given. Thus by definition of the other differences, the just imperceptible negative difference implies that all the smaller stimuli were judged lighter but this one was judged not-lighter. The just perceptible positive difference implies that all smaller stimuli were judged not-heavier while on this one a heavier judgment was passed. Finally the just imperceptible positive difference implies that, on all greater stimuli, a heavier judgment was passed but on this one a not-heavier. Now let q be the probability of a lighter judgment on a given intensity of the comparison stimulus. Then by definition, the probability of a not-lighter judgment will be 1 — q, as these probabilities are mutually exclusive. In this same way 1 — p will be the probability of a not-heavier judgment, if the probability of a heavier judgment is p Now suppose we have a series of comparison stimuli r_1, r_2, . . . r_n arranged in the order of their intensity, in which r_1 is the heaviest, and we use this series for the determination of the just perceptible negative difference. Let us designate the probabilities of a lighter judgment on the first, second, . . . n comparison stimuli by q_1, q_2, . . . q_n. There exists for each stimulus a certain probability that it will be obtained as a determination of the just perceptible negative difference, which we call Q_1, Q_2, . . . Q_n.

$$Q_1 = q_1 ;$$
$$Q_2 = (1-q_1)q_2 ;$$
$$Q_{n-1} = (1-q_1)(1-q_2) \dots\dots (1-q_{n-2})q_{n-1} ;$$
$$Q_n = (1-q_1)(1-q_2) \dots\dots (1-q_{n-1})q_n .$$

The probability that a stimulus will be obtained as a determination of the just imperceptible negative difference can be similarly analyzed. For the comparison stimuli r_1, r_2 . . . r_n we have

$$Q^1_1 = q_n \cdot q_{n-1} \cdot \cdots \cdots q_2 (1-q_1) \; ;$$
$$Q^1_2 = q_n \cdot q_{n-1} \cdot \cdots \cdots q_3 (1-q_2) \; ;$$
$$Q^1_{n-1} = q_n (1-q_{n-1}) \; ;$$
$$Q^1_n = (1-q_{n-1}) \; .$$

The probabilities that the just perceptible positive difference will fall on the stimuli r_1, r_2, . . . r_n are expressed

$$P_1 = (1-p_n)(1-p_{n-1}) \cdots \cdots (1-p_2) p_1 \; ;$$
$$P_2 = (1-p_n)(1-p_{n-1}) \cdots \cdots (1-p_3) p_2 \; ;$$
$$P_{n-1} = (1-p_n) p_{n-1} \; ;$$
$$P_n = p_n \; .$$

Finally the probabilities with which the different stimuli will be obtained as a determination of the just imperceptible positive difference for the stimuli r_1, r_2, . . . r_n are

$$P^1_1 = (1-p_1) \; ;$$
$$P^1_2 = p_1 (1-p_2) \; ;$$
$$P^1_{n-1} = p_1 \cdot p_2 \cdots \cdots p_{n-2} (1-p_{n-1}) \; ;$$
$$P^1_n = p_1 \cdot p_2 \cdots \cdots p_{n-1} (1-p_n) \; .$$

We now turn to a discussion of the results of our experiments. It will be remembered that one of the comparison stimuli was changed systematically so that the results on it could serve as a determination of the thresholds of the method of just perceptible differences, and at the same time, the judgments on the other comparison stimuli enabled us to apply the method of constant stimuli; the relation of these two methods being the principal problem of this paper. Obviously there is no difference in the form of the individual judgments of these two methods, since they consist in the passing of a judgment of whether the comparison stimulus is subjectively lighter, equal or heavier than the standard stimulus. Our results of the method of just perceptible differences are of two forms; observed values that were taken simultaneously with those by the method of constant stimuli, and calculated values. One way of obtaining these calculated values

	I	II	III	IV	V	VI	VII	VIII	IX	X	Total
84.80		10	10	10	10	10	10	10	10	10	100
85.22											10
85.64		10									10
86.06			10	10							20
86.48		10			10						10
86.90						10					30
87.32		10	10				10				20
87.74	10			10				10			30
88.16		10							10		20
88.58						10				10	30
89.00		10		10		10					40
89.84		10	10								10
90.26	10						10				20
90.68		10			10						20
91.10			10	10				10			30
91.52		10				10			10		50
92.36	10	10	10	10	10					10	40
93.20		10					10				20
93.62			10								20
94.04	10	10		10		10		10			50
94.88		10	10		10						20
95.30	10										10
95.72		10							10		30
96.14	10		10	10			10				40
96.56	10	10									10
96.98	10				10	10				10	60
97.40	10	10									10
97.82	10			10				10			40
98.24	10	10									20
98.66	10		10								10
99.08		10			10		10				30
99.50	10			10		10			10		50
99.92		10	10								20
100.76	10	10									10
101.18				10	10			10		10	50
101.60		10				10	10				50
102.44	10	10	10	10							20
103.28		10			10				10		30
103.70			10								10
104.12		10									10
104.54	10			10		10		10			50
104.96		10	10				10				10
105.38					10					10	30
105.80		10									10
106.22			10	10							20
106.64		10									10
107.06	10					10			10		40
107.48		10	10		10		10				10
107.90				10				10			40
108.32		10									10
108.74			10								
110.00	10					10				10	30
111.26							10		10		20
112.52						10					10

is that employed by Urban [*Stat. Meth.*], where the experimental data were in such a form that it was possible to apply both forms of mathematical treatment to the same set of results. This is not possible in the present study as a glance at the table below will convince one that not enough judgments were taken on each intensity of the comparison stimulus to enable us to apply the treatment of the method of constant stimuli. In this table are found the number of judgments passed on the intensities of the comparison stimulus for each series, and the last column gives the totals for each intensity used.

The other way of approaching this problem is to take the results of the method of constant stimuli and calculate from them the results we are likely to obtain from the method of just perceptible differences. This is possible because the determination of the quantities h and c determines the entire course of the psychometric functions. We, therefore, are able to obtain the probabilities of all three judgments on any comparison stimulus, if the constants h and c for the lighter and heavier judgments are given. These probabiltiies are indeed, all that we need for the calculation of the method of just perceptible differences, and we therefore can find the most probable value of any one of the four differences. The conditions under which our results for the methods of just perceptible differences and constant stimuli were taken, were very much the same, if not entirely alike, and we therefore, may expect to obtain calculated results which agree with the observed results within the limits of accuracy obtainable in this kind of experiments. The calculation of the probabilities with which the different intensities of the comparison stimulus will be obtained as determinations of the different perceptible and imperceptible positive and negative differences is easy but rather long and somewhat complicated. So it is necessary to plan the whole work systematically in a manner which will now be described.

The first step consisted in the collection of the material for the determination of the constants of the psychometric functions. The records, as they were taken during experimentation, were entered in tables, similar in form to those described above as the records of first entry, but the records of each series of the method of just perceptible differences were kept separate. The relative frequencies of the judgments in this division of the results by series, are found in tables XIV and XV for subjects I and II respectively. These are similar in form to tables III and IV, with the exception that the numbers in the first column, instead of representing groups of 100 judgments, as in the former calculation, now indicate the series of the method of just perceptible differences with which they were taken simultaneously. As has been pointed out above, the length of the different series of the method of just perceptible differences varied; there being only seven judgments for series X; while series II required the passing of twenty-nine judgments. Thus it is obvious that the number of judgments from which these relative frequencies were calculated vary in a similar ratio.

From these relative frequencies, the constants for the method of constant stimuli, h and c, were calculated by the same method that has been described above. In order to show the steps of this calculation, tables XVI and XVII give, for the lighter and heavier judgments of subject I, the values of [P], [xP], [xxP], [γP], [xγP], [sP] and [xsP]. The corresponding values are found for subject II in tables XVIII and XIX. These four tables are similar in form to tables VII-X, which have been described above. The numbers in the first five columns give the coefficients for setting up the normal equations and the data of the columns [sP] and [xsP] show that all the necessary checks are fulfilled. Tables XX and XXI, which are similar in form to tables XI and XII, give the constants h and c for the lighter and heavier judgments for both subjects. The thresholds S_1 and S_2 are obtained in the same way as in the former calculation and are included in these tables. The series of the method of just perceptible differences were performed in haphazard order, so that the results that go to make up the values included in these tables, are from various parts of the experiment at different stages of practice, and we, therefore, must not look for the effect of practice in these constants. In fact the results that were taken at the end of the experiment in each series, where the practice was high, tend to cancel any

Series	84			88			92			96			104			108		
	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier
I.........	.963	.032	.005	.895	.105	.000	.626	.263	.111	.458	.363	.179	.063	.158	.779	.032	.063	.905
II.........	.966	.034	.000	.876	.114	.010	.659	.210	.131	.441	.359	.200	.072	.152	.776	.024	.055	.921
III.........	.960	.040	.000	.880	.100	.020	.600	.275	.125	.405	.400	.195	.085	.140	.775	.010	.060	.930
IV.........	.973	.020	.007	.907	.073	.020	.667	.220	.113	.353	.407	.240	.087	.226	.687	.013	.073	.914
V.........	.950	.042	.008	.867	.100	.033	.675	.250	.075	.350	.367	.283	.050	.175	.775	.000	.042	.958
VI.........	.983	.017	.000	.833	.150	.017	.583	.300	.117	.467	.333	.200	.075	.133	.792	.008	.083	.908
VII.........	.930	.050	.020	.850	.140	.010	.650	.210	.140	.410	.440	.150	.020	.110	.870	.010	.050	.940
VIII.........	.988	.012	.000	.825	.125	.050	.650	.250	.100	.500	.338	.162	.038	.150	.812	.012	.038	.950
IX.........	.988	.012	.000	.875	.125	.000	.737	.200	.063	.375	.350	.275	.038	.162	.800	.012	.088	.900
X.........	.971	.029	.000	.886	.114	.000	.600	.314	.086	.286	.443	.271	.057	.114	.829	.014	.000	.986

TABLE XIV

Series	84			88			92			96			104			108		
	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier	Lighter	Equal	Heavier
I.........	.989	.011	.000	.795	.173	.032	.579	.295	.126	.257	.417	.326	.058	.163	.799	.016	.042	.942
II.........	.962	.035	.003	.859	.100	.041	.607	.245	.148	.417	.269	.314	.048	.155	.797	.026	.059	.915
III.........	.955	.025	.020	.860	.120	.020	.585	.225	.190	.385	.270	.345	.070	.125	.805	.000	.040	.960
IV.........	.973	.020	.007	.753	.167	.080	.640	.213	.147	.380	.287	.330	.033	.120	.847	.020	.060	.920
V.........	.925	.050	.025	.775	.200	.025	.575	.333	.092	.375	.342	.283	.033	.134	.833	.008	.050	.942
VI.........	.917	.067	.016	.800	.183	.017	.533	.225	.242	.358	.308	.334	.058	.100	.842	.025	.092	.883
VII.........	.960	.030	.010	.900	.080	.020	.610	.280	.110	.340	.350	.310	.030	.100	.870	.010	.090	.900
VIII.........	.963	.037	.000	.865	.085	.050	.600	.225	.175	.375	.412	.213	.075	.125	.800	.012	.050	.938
IX.........	.962	.025	.013	.900	.075	.025	.600	.288	.112	.400	.362	.238	.025	.075	.900	.013	.012	.975
X.........	.986	.014	.000	.886	.071	.043	.614	.243	.143	.414	.286	.300	.014	.172	.814	.029	.071	.900

TABLE XV

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
I	3.467	327.50	31083.04	— 0.215	— 3.890	324.244	30759.43
II	3.263	308.27	29264.83	— 0.201	— 3.053	305.210	28959.61
III	3.456	324.72	30639.88	— 0.220	— 5.965	321.480	30321.13
IV	3.260	307.92	29209.52	— 0.138	1.943	304.802	28899.65
V	3.207	297.73	27736.32	— 0.534	—37.912	295.055	27476.50
VI	3.196	298.37	27946.14	— 0.406	—27.664	295.578	27675.41
VII	3.332	308.34	28643.10	— 0.590	—41.793	305.602	28376.55
VIII	3.152	295.88	27873.49	— 0.325	—18.834	293.057	27596.44
IX	2.981	280.20	26434.32	— 0.297	—15.127	277.520	26169.24
X	3.213	301.77	28461.02	0.012	15.857	298.543	28143.40

Table XVI

Series	[P]	[xP]	[xxP]	[γP]	[xγP]	[sP]	[sxP]
I	2.683	267.31	26741.18	— 0.152	— 2.078	264.781	26475.95
II	2.782	275.77	27445.62	— 0.235	—10.051	273.225	27179.89
III	2.801	276.67	27442.42	— 0.342	—19.806	274.209	27185.55
IV	3.081	304.28	30189.97	— 0.524	—36.269	301.727	29921.95
V	2.820	275.94	27126.43	— 0.505	—33.952	273.629	26884.42
VI	2.773	275.27	27437.04	— 0.228	— 8.872	272.723	27170.64
VII	2.632	256.67	25164.94	— 0.510	—33.105	254.546	24941.38
VIII	2.688	263.29	25908.38	— 0.506	—34.691	261.106	25679.78
IX	2.593	259.91	26139.30	0.133	24.084	257.186	25855.30
X	2.108	208.28	20634.69	— 0.038	4.275	206.210	20422.13

Table XVII

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
I.........	3.279	308.18	29080.22	0.281	39.716	304.616	28732.33
II.........	3.477	326.78	30852.98	— 0.178	— 0.928	323.481	30527.12
III.........	3.362	313.20	29281.58	— 0.382	—23.776	310.216	28992.16
IV.........	3.446	321.88	30188.08	— 0.137	0.914	318.567	29865.29
V.........	3.570	330.05	30631.86	— 0.310	—15.647	326.792	30317.45
VI.........	3.807	355.25	33313.87	— 0.010	15.666	351.455	32942.95
VII.........	3.115	290.54	27200.90	— 0.266	—11.524	287.687	26921.89
VIII.........	3.443	323.25	30477.95	— 0.151	0.699	319.956	30154.00
IX.........	3.138	292.88	27439.57	— 0.338	—18.450	290.084	27165.13
X.........	3.058	287.41	27117.62	— 0.214	— 7.486	284.568	26837.62

Table XVIII

Series	[P]	[xP]	[xxP]	[γP]	[γxP]	[sP]	[sxP]
I.........	2.988	293.38	28924.10	— 0.267	—12.282	290.655	28642.99
II.........	3.190	312.77	30807.74	— 0.322	—15.760	309.900	30510.74
III.........	3.134	303.85	29600.05	— 0.437	—26.432	301.155	29322.63
IV.........	3.293	319.85	31225.86	— 0.427	—24.070	316.982	30930.08
V.........	2.928	284.98	27886.22	— 0.534	—34.247	282.586	27635.47
VI.........	3.366	329.17	32354.14	— 0.200	— 2.893	326.002	32027.87
VII.........	2.939	288.46	28452.88	— 0.286	—11.280	285.807	28175.70
VIII.........	3.041	297.24	29185.01	— 0.435	—27.659	294.630	28915.43
IX.........	2.511	242.32	23492.45	— 0.637	—46.675	240.446	23296.80
X.........	3.157	310.32	30643.18	— 0.257	—10.028	307.424	30342.88

Table XIX

Subject I

Series	Lighter Judgments			Heavier Judgments		
	h_1	c_1	S_1	h_2	c_2	S_2
I	0.11214	10.655	95.02	0.12064	12.076	100.10
II	0.11266	10.705	95.02	0.12071	12.050	99.82
III	0.11354	10.732	94.52	0.12315	12.287	99.77
IV	0.11883	11.266	94.81	0.11088	11.121	100.30
V	0.12186	11.480	94.20	0.12373	12.286	99.30
VI	0.11265	10.644	94.48	0.12340	12.332	99.93
VII	0.11650	10.958	94.06	0.12359	12.246	99.09
VIII	0.11833	11.211	94.74	0.12528	12.459	99.45
IX	0.13213	12.519	94.75	0.12345	12.323	99.82
X	0.12429	11.670	93.89	0.14430	14.276	98.93

Table XX

Subject II

Series	Lighter Judgments			Heavier Judgments		
	h_1	c_1	S_1	h_2	c_2	S_2
I	0.11824	11.027	93.26	0.11783	11.659	98.94
II	0.11210	10.587	94.44	0.11134	11.018	98.96
III	0.11295	10.636	94.16	0.11297	11.092	98.19
IV	0.11189	10.491	93.76	0.10936	10.752	98.32
V	0.11007	10.263	93.24	0.11875	11.740	98.86
VI	0.10147	9.472	93.34	0.10204	10.038	98.37
VII	0.13067	12.273	93.93	0.11900	11.777	98.96
VIII	0.11490	10.831	94.27	0.11320	11.208	99.00
IX	0.12538	11.810	94.20	0.13827	13.597	98.34
X	0.12024	11.371	94.57	0.10909	10.805	99.04

Table XXI

variation of the results at the beginning, where practice was low. Thus these values of the threshold show remarkably little variation. The numbers under the headings S_1 and S_2 give the thresholds in the direction of increase and decrease by the method of constant stimuli, for the group of experiments that were made simultaneously with the corresponding series of the method of just perceptible differences.

After having obtained the constants h and c, we can ascertain the probability of a heavier or lighter judgment for any intensity of the comparison stimulus, by the formula

$$p = \tfrac{1}{2} \pm \tfrac{1}{2}\,\Phi(\gamma)\ .$$

This calculation is made in two steps; as first γ must be calculated by the formula

$$\gamma = hx - c$$

in which x is the value of the intensity of the stimulus for which we are calculating the probability. The value of $\Phi(\gamma)$ is then looked in a table of the probability integral [Czuber. *Wahr-*

scheinlichkeitsrechnung, 1908, pp. 388-391]. This value of $\Phi(\gamma)$, which takes the sign of its argument, is substituted in the formula given above and we obtain the probability of that judgment for the weight x. These values must be found for every intensity of the comparison stimulus that was used in the corresponding series for the method of just perceptible differences.

In actual practice this calculation is very much simplified. If we desire the probabilities of the lighter judgments q, for series II, we first find γ_1 for $x = 84.80$, this being our lightest comparison weight. We then find the value of γ for x equal to the amount of difference of the steps of this series. This value is subtracted from γ_1 and we obtain γ_2 or that of the next heavier comparison stimulus. Then γ is subtracted from γ_2 and we obtain γ_3 for the next heavier comparison weight, and so forth. As our γ values are only carried out to four places, corrections have to be made when necessary. A check on this step of the calculations is effected by finding the γ of the heaviest comparison weight by means of the formula, and ascertaining if it coincides with the value obtained by the series of subtractions. These γ's are then successively introduced into the formula for finding the probabilities, given above, and the different probabilities of the lighter judgment — q — are obtained for all of the comparison stimuli used. The probabilities that a not-lighter—equal or heavier—judgment will be passed are obtained by subtracting the probabilities of a lighter judgment, in each case, from unity.

From this series of the q's and (I — q)'s we may obtain the just perceptible negative difference and the just imperceptible negative difference, by means of the formulae given above. This calculation is much simplified by the use of logarithms. These are looked up for every value of q and 1 — q. Then the probability that our lightest weight will be obtained as a determination of the just imperceptible negative difference is $Q_1 = (1 - q_1)$ and is found directly. Adding log q_1 to log $(1 - q_2)$ gives log Q_2 which is the probability that our next heavier comparison stimulus will be obtained as a determination of this difference. Then log q_1 is added to log q_2 and to their sum is added log $(1 - q_3)$ which gives Q_3. To the sum of log q_1 and q_2 is added log q_3 and to this sum is added log $(1 - q_4)$, which gives the probability that this difference will fall on the fourth comparison weight. This scheme of calculation is very simple and is learned rapidly; its mechanism will be easily understood from the following example, where the formulae are found alongside part of one of our calculations.

$\log Q_1 =$	$\log q_1$	$= .94448 - 3$
	$\log (1-q_1)$	$= .99616 - 1$
	$\log (q_2)$	$= .44716 - 2$
$\log Q_2 =$	$\log (1-q_1)\, q_2$	$= .44332 - 2$
	$\log (1-q_1)$	$= .99616 - 1$
	$\log (1-q_2)$	$= .98767 - 1$
	$\log (1-q_1)(1-q_2)$	$= .98383 - 1$
	$\log q_3$	$= .81558 - 2$
$\log Q_3 =$	$\log (1-q_1)(1-q_2)\, q_3$	$= .79941 - 2$
	$\log (1-q_1)(1-q_2)$	$= .98383 - 1$
	$\log (1-q_3)$	$= .97063 - 1$
	$\log (1-q_1)(1-q_2)(1-q_3)$	$= .95446 - 1$
	$\log q_4$	$= .07700 - 1$
$\log Q_4 =$	$\log (1-q_1)(1-q_2)(1-q_3)\, q_4$	$= .03146 - 1$
	.	.
	.	.
	.	.
	.	.
	.	.

In this way the calculation is reduced to a number of successive additions by carrying the sums of the terms that accumulate. These additions give us the values of Q and Q_1, or the probabilities with which the just perceptible and imperceptible negative differences will fall on the different values of the comparison stimulus. A similar calculation with the h and c for the heavier judgments gives the values of P and P_1, or the probabilities that the different intensities of the comparison stimulus will be found as a determination of the just perceptible and imperceptible positive differences. Both sets of calculations for the heavier and lighter judgments, had to be performed for each of the ten series for both subjects.

A check on this part of the calculations is effected by the fact that, as these are probabilities of mutually exclusive events, all the values in one set must add up to unity. This had to prove within a limit of 0.0001, which was the exactitude of our calculations. It was found for the longer series that this calculation did not have to be carried through for all of the values of x, as frequently the probabilities of the differences became so small as to no longer affect the last decimal place. In the short series, on the other hand, it frequently happened that there was a remainder, and this had to be calculated as it was necessary for a check of the correctness of our work. This remainder has some significance since it gives the probability that the series of comparison stimuli r_1, r_2, . . . r_n will be gone through without giving a determination of the quantity sought for. Thus the remainder for the just perceptible positive difference gives the probability that no stimulus of the series will be obtained as a determination of this quantity or—what is the same—that no heavier judgment will be given on any of the stimuli used. This remainder has the form of a product of probabilities and obviously becomes very small if any factors are small. Thus in our short series there was greater likelihood of having larger individual probabilities and we find a tendency for greater remainders to occur in the short series rather than in the long ones. This is the reason of the rule that the series of comparison stimuli should be extended so far that we will have a very high probability of obtaining only heavier judgments on the largest comparison stimulus, and only lighter judgments on the smallest stimulus used. It is, therefore, not desirable to work with series in which the remainders are of at all considerable size. None of the remainders of our series are large enough to invalidate our results in any way.

Tables XXII-XXXI give the results of these calculations for

Series I

Comparison Stimulus	q	Q	Q^1	p	P	P^1
84.80	0.947	0.0001	0.0526	0.005	0.0046	
87.74	0.876	0.0003	0.1177	0.018	0.0174	
90.26	0.774	0.0014	0.1871	0.047	0.0456	
92.36	0.663	0.0034	0.2164	0.093	0.0869	
94.04	0.561	0.0066	0.1870	0.151	0.1273	
95.30	0.482	0.0110	0.1239	0.206	0.1482	0.0001
96.14	0.429	0.0172	0.0658	0.250	0.1422	0.0001
96.56	0.403	0.0270	0.0295	0.273	0.1168	0.0004
96.98	0.378	0.0406	0.0124	0.297	0.0925	0.0013
97.40	0.353	0.0586	0.0049	0.322	0.0705	0.0038
97.82	0.328	0.0809	0.0018	0.349	0.0516	0.0106
98.24	0.304	0.1082	0.0006	0.376	0.0362	0.0269
98.66	0.282	0.1392	0.0002	0.403	0.0242	0.0640
99.50	0.234	0.1548	0.0001	0.459	0.0165	0.1261
100.76	0.181	0.1437		0.545	0.0106	0.1948
102.44	0.119	0.1075		0.655	0.0058	0.2255
104.54	0.065	0.0630		0.776	0.0024	0.1889
107.06	0.028	0.0278		0.883	0.0006	0.1120
110.00	0.009	0.0088		0.954	0.0001	0.0456
Σ		1.0001	1.0000		1.0000	1.0001
R		0.0000	0.0000		0.0000	0.0000

Table XXII

Series II

Comparison Stimulus	q	Q	Q^1	p	P	P^1
84.80	0.948		0.0518	0.005	0.0052	
85.64	0.932		0.0641	0.008	0.0077	
86.48	0.913		0.0767	0.013	0.0131	
87.32	0.890		0.0886	0.016	0.0160	
88.16	0.863		0.0986	0.023	0.0222	
89.00	0.831		0.1047	0.032	0.0301	
89.84	0.796	0.0001	0.1054	0.044	0.0398	
90.68	0.755	0.0002	0.1003	0.059	0.0513	
91.52	0.712	0.0007	0.0893	0.078	0.0635	
92.36	0.664	0.0020	0.0740	0.101	0.0761	
93.20	0.614	0.0048	0.0565	0.129	0.0871	
94.04	0.562	0.0101	0.0394	0.161	0.0949	
94.88	0.509	0.0187	0.0248	0.199	0.0982	0.0001
95.72	0.456	0.0308	0.0140	0.242	0.0953	0.0004
96.56	0.403	0.0456	0.0070	0.288	0.0863	0.0014
97.40	0.352	0.0615	0.0031	0.339	0.0723	0.0039
98.24	0.304	0.0764	0.0012	0.394	0.0554	0.0090
99.08	0.259	0.0877	0.0004	0.449	0.0384	0.0182
99.92	0.218	0.0944	0.0001	0.506	0.0238	0.0322
100.76	0.180	0.0953		0.563	0.0131	0.0507
101.60	0.147	0.0914		0.618	0.0063	0.0716
102.44	0.119	0.0836		0.672	0.0026	0.0915
103.28	0.094	0.0732		0.722	0.0009	0.1076
104.12	0.074	0.0619		0.768	0.0003	0.1167
104.96	0.057	0.0505		0.809	0.0001	0.1186
105.80	0.043	0.0399		0.846		0.1132
106.64	0.032	0.0309		0.878		0.1026
107.48	0.024	0.0232		0.904		0.0886
108.32	0.017	0.0170		0.926		0.0736
Σ		0.9999	1.0000		1.0000	0.9999
R		0.0000	0.0000		0.0000	0.0000

Table XXIII

Series III

Comparison Stimulus	q	Q	Q'	p	P	P'
84.80	0.941		0.0592	0.005	0.0046	
86.06	0.913		0.0819	0.008	0.0084	
87.32	0.876	0.0001	0.1062	0.015	0.0148	
88.58	0.830	0.0007	0.1279	0.026	0.0249	
89.84	0.774	0.0030	0.1412	0.042	0.0396	
91.10	0.709	0.0095	0.1408	0.065	0.0594	
92.36	0.636	0.0233	0.1248	0.099	0.0836	0.0001
93.62	0.557	0.0461	0.0965	0.142	0.1088	0.0006
94.88	0.477	0.0754	0.0636	0.197	0.1294	0.0029
96.14	0.397	0.1043	0.0349	0.264	0.1388	0.0100
97.40	0.322	0.1245	0.0156	0.340	0.1318	0.0263
98.66	0.253	0.1312	0.0055	0.423	0.1083	0.0543
99.92	0.193	0.1239	0.0015	0.510	0.0753	0.0904
101.18	0.142	0.1067	0.0003	0.596	0.0431	0.1251
102.44	0.102	0.0848	0.0001	0.678	0.0198	0.1470
103.70	0.070	0.0630		0.752	0.0071	0.1505
104.96	0.047	0.0441		0.816	0.0019	0.1367
106.22	0.030	0.0293		0.869	0.0004	0.1122
107.48	0.019	0.0186		0.910	0.0001	0.0847
108.74	0.011	0.0114		0.941		0.0592
Σ		0.9999	1.0000		1.0001	1.0000
R		0.0000	0.0000		0.0000	0.0000

Table XXIV

Series IV

Comparison Stimulus	q	Q	Q'	p	P	P'
84.80	0.954		0.0464	0.008	0.0076	
86.48	0.919	0.0004	0.0770	0.015	0.0151	
88.16	0.868	0.0026	0.1157	0.028	0.0278	
89.84	0.798	0.0121	0.1537	0.051	0.0480	
91.52	0.710	0.0370	0.1762	0.084	0.0761	0.0003
93.20	0.606	0.0803	0.1695	0.133	0.1096	0.0019
94.88	0.495	0.1297	0.1320	0.198	0.1415	0.0091
96.56	0.384	0.1634	0.0796	0.279	0.1603	0.0293
98.24	0.282	0.1672	0.0357	0.374	0.1547	0.0680
99.92	0.195	0.1434	0.0113	0.476	0.1235	0.1194
101.60	0.137	0.1166	0.0024	0.581	0.0788	0.1646
103.28	0.077	0.0713	0.0003	0.680	0.0387	0.1851
104.96	0.044	0.0424		0.768	0.0140	0.1749
106.64	0.023	0.0231		0.840	0.0036	0.1433
108.32	0.011	0.0106		0.896	0.0006	0.1041
Σ		1.0001	0.9999		0.9999	1.0000
R		0.0000	0.0000		0.0001	0.0000

Table XXV

Series V

Comparison Stimulus	q	Q	Q^1	p	P	P^1
84.80	0.947	0.0006	0.0526	0.006	0.0056	
86.90	0.896	0.0059	0.0986	0.015	0.0149	
89.00	0.815	0.0288	0.1570	0.036	0.0351	0.0002
91.10	0.703	0.0837	0.2055	0.076	0.0716	0.0025
93.20	0.568	0.1566	0.2101	0.143	0.1250	0.0161
95.30	0.425	0.2040	0.1587	0.242	0.1813	0.0586
97.40	0.291	0.1970	0.0833	0.370	0.2099	0.1315
99.50	0.181	0.1493	0.0280	0.514	0.1832	0.1978
101.60	0.101	0.0928	0.0056	0.656	0.1138	0.2133
103.70	0.051	0.0494	0.0006	0.779	0.0465	0.1757
105.80	0.023	0.0227		0.852	0.0112	0.1380
107.90	0.009	0.0092		0.934	0.0018	0.0662
Σ		1.0000	1.0000		0.9999	0.9999
R		0.0000	0.0000		0.0001	0.0000

TABLE XXVI

Series VI

Comparison Stimulus	q	Q	Q^1	p	P	P^1
84.80	0.939	0.0024	0.0610	0.004	0.0040	
87.32	0.873	0.0176	0.1193	0.019	0.0189	0.0001
89.84	0.770	0.0677	0.1885	0.039	0.0381	0.0013
92.36	0.632	0.1509	0.2323	0.093	0.0873	0.0131
94.88	0.475	0.2161	0.2094	0.189	0.1610	0.0619
97.40	0.321	0.2151	0.1287	0.329	0.2272	0.1556
99.92	0.193	0.1602	0.0490	0.499	0.2313	0.2328
102.44	0.103	0.0953	0.0105	0.669	0.1553	0.2299
104.96	0.048	0.0466	0.0012	0.810	0.0623	0.1629
107.48	0.019	0.0188	0.0001	0.906	0.0132	0.0890
110.00	0.007	0.0070		0.960	0.0013	0.0394
112.52	0.002	0.0020		0.986	0.0001	0.0140
Σ		0.9997	1.0000		1.0000	1.0000
R		0.0002	0.0000		0.0000	0.0000

TABLE XXVII

Series VII

Comparison Stimulus	q	Q	Q^1	p	P	P^1
84.80	0.936	0.0090	0.0635	0.006	0.0062	
87.74	0.851	0.0546	0.1395	0.024	0.0235	0.0012
90.68	0.711	0.1581	0.2301	0.070	0.0684	0.0158
93.62	0.529	0.2494	0.2672	0.168	0.1519	0.0837
96.56	0.340	0.2434	0.1977	0.328	0.2456	0.2067
99.50	0.185	0.1623	0.0831	0.529	0.2670	0.2734
102.44	0.084	0.0802	0.0173	0.721	0.1713	0.2243
105.38	0.031	0.0307	0.0015	0.865	0.0572	0.1260
108.32	0.009	0.0094	0.0001	0.947	0.0085	0.0523
111.26	0.002	0.0023		0.983	0.0005	0.0166
Σ		0.9994	1.0000		1.0001	1.0000
R		0.0006	0.0000		0.0000	0.0000

TABLE XXVIII

Series VIII

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.952	0.0116	0.0480	0.005	0.0048	0.0001
88.16	0.865	0.0775	0.1288	0.023	0.0227	0.0035
91.52	0.704	0.2137	0.2433	0.080	0.0779	0.0409
94.88	0.491	0.2927	0.2952	0.209	0.1872	0.1678
98.24	0.279	0.2308	0.2053	0.416	0.2943	0.2979
101.60	0.126	0.1189	0.0695	0.648	0.2678	0.2768
104.96	0.043	0.0427	0.0096	0.835	0.1214	0.1551
108.32	0.012	0.0116	0.0004	0.942	0.0225	0.0580
Σ		0.9995	1.0001		0.9986	1.0001
R		0.0006	0.0000		0.0014	0.0000

TABLE XXIX

Series IX

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.968	0.0187	0.0316	0.004	0.0044	0.0002
88.58	0.876	0.1358	0.1203	0.025	0.0249	0.0084
92.36	0.673	0.3192	0.2773	0.097	0.0940	0.0801
96.14	0.397	0.3120	0.3443	0.262	0.2295	0.2500
99.92	0.167	0.1573	0.1887	0.507	0.3280	0.3295
103.70	0.047	0.0468	0.0360	0.752	0.2400	0.2203
107.48	0.009	0.0086	0.0018	0.909	0.0720	0.0886
111.26	0.001	0.0010		0.977	0.0070	0.0229
Σ		0.9994	1.0000		0.9998	1.0000
R		0.0006	0.0000		0.0001	0.0000

TABLE XXX

Series X

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.945	0.0544	0.0550	0.002	0.0020	0.0006
89.00	0.805	0.2375	0.1842	0.021	0.0214	0.0287
93.20	0.548	0.3583	0.3435	0.121	0.1183	0.2130
97.40	0.269	0.2402	0.3050	0.377	0.3239	0.3999
101.60	0.088	0.0860	0.1023	0.707	0.3779	0.2662
105.80	0.018	0.0182	0.0097	0.919	0.1440	0.0796
110.00	0.002	0.0024	0.0002	0.988	0.0125	0.0120
Σ		0.9970	0.9999		1.0000	1.0000
R		0.0031	0.0000		0.0001	0.0000

TABLE XXXI

subject I; one table being given to each of the ten series [I-X]. In the first columns are found the intensities of the comparison stimuli used in each particular series. The second columns give the probabilities of a lighter judgment on these comparison stimuli. The third and fourth columns give, under the headings Q and Q_1, the probabilities with which the different comparison stimuli appear as determinations of the just perceptible and of the just imperceptible negative difference. The next column gives the probabilities with which heavier judgments may be expected on the stimuli. These last values serve as a basis for the calculation of the probabilities of the different stimuli for being observed as determinations of the just perceptible and of the just imperceptible positive difference, the values of which are found in the last columns of the table. The numbers at the bottom of each column—marked Σ—give the sums of all of the terms and the numbers R give the values of the remainder. These remainders and the sums of the probabilities when added up come to unity within the limit of 0.0001 and thus prove the correctness of the computation. Tables XXXII-XLI are similarly constructed and give the corresponding values of subject II.

It will be noticed that the size of the remainder is in no case so large as to invalidate the series. There is a tendency for the size of the remainder to increase as the series decreases in length. A glance at the table that follows, which contains all the remainders for both subjects, shows this tendency. In the first four

Series I (G)

Comparison Stimulus	q	Q	Q^1	p	P	P^1
84.80	0.921	0.0007	0.0786	0.010	0.0100	
87.74	0.822	0.0036	0.1642	0.031	0.0305	
90.26	0.692	0.0099	0.2331	0.074	0.0712	
92.36	0.560	0.0183	0.2308	0.136	0.1210	
94.04	0.448	0.0265	0.1618	0.207	0.1587	0.0001
95.30	0.367	0.0342	0.0833	0.272	0.1653	0.0001
96.14	0.315	0.0429	0.0330	0.320	0.1418	0.0006
96.56	0.291	0.0558	0.0108	0.346	0.1042	0.0016
96.98	0.267	0.0699	0.0032	0.371	0.0733	0.0040
97.40	0.244	0.0847	0.0009	0.398	0.0494	0.0097
97.82	0.223	0.0995	0.0002	0.425	0.0317	0.0218
98.24	0.202	0.1132	0.0001	0.453	0.0195	0.0457
98.66	0.183	0.1254		0.481	0.0113	0.0902
99.50	0.148	0.1192		0.532	0.0065	0.1501
100.76	0.105	0.0941		0.619	0.0035	0.1995
102.44	0.062	0.0596		0.720	0.0015	0.2036
104.54	0.030	0.0292		0.824	0.0005	0.1550
107.06	0.010	0.0105		0.912	0.0001	0.0853
110.00	0.003	0.0026		0.967		0.0328
Σ		0.9998	1.0000		1.0000	1.0001
R		0.0001	0.0000		0.0000	0.0000

TABLE XXXII

Series II

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.938		0.0621	0.013	0.0130	
85.64	0.919		0.0763	0.018	0.0178	
86.48	0.897		0.0890	0.025	0.0240	
87.32	0.871		0.0998	0.033	0.0316	
88.16	0.841		0.1072	0.045	0.0408	
89.00	0.806	0.0001	0.1098	0.058	0.0510	
89.84	0.767	0.0001	0.1061	0.076	0.0621	
90.68	0.725	0.0005	0.0963	0.096	0.0731	
91.52	0.679	0.0015	0.0814	0.121	0.0829	
92.36	0.629	0.0037	0.0638	0.149	0.0901	
93.20	0.578	0.0080	0.0457	0.182	0.0937	
94.04	0.525	0.0153	0.0297	0.219	0.0921	0.0001
94.88	0.472	0.0261	0.0173	0.260	0.0854	0.0004
95.72	0.420	0.0400	0.0090	0.305	0.0739	0.0012
96.56	0.368	0.0555	0.0041	0.353	0.0595	0.0031
97.40	0.319	0.0708	0.0016	0.403	0.0439	0.0071
98.24	0.273	0.0834	0.0006	0.455	0.0296	0.0141
99.08	0.231	0.0916	0.0002	0.508	0.0180	0.0252
99.92	0.193	0.0946	0.0001	0.559	0.0098	0.0404
100.76	0.158	0.0922		0.612	0.0047	0.0581
101.60	0.128	0.0856		0.661	0.0020	0.0766
102.44	0.102	0.0762		0.708	0.0007	0.0931
103.28	0.081	0.0653		0.752	0.0002	0.1054
104.12	0.062	0.0539		0.796	0.0001	0.1089
104.96	0.048	0.0433		0.828		0.1112
105.80	0.036	0.0337		0.859		0.1055
106.64	0.027	0.0256		0.887		0.0959
107.48	0.019	0.0191		0.910		0.0835
108.32	0.014	0.0139		0.930		0.0703
Σ		1.0000	1.0001		1.0000	1.0001
R		0.0000	0.0000		0.0000	0.0000

Table XXXIII

Series III

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.933		0.0673	0.016	0.0162	
86.06	0.902		0.0910	0.026	0.0260	
87.32	0.863	0.0002	0.1155	0.041	0.0395	
88.58	0.814	0.0011	0.1351	0.062	0.0573	
89.84	0.755	0.0042	0.1446	0.091	0.0785	
91.10	0.688	0.0123	0.1394	0.129	0.1006	0.0002
92.36	0.614	0.0284	0.1187	0.176	0.1200	0.0009
93.62	0.535	0.0533	0.0876	0.233	0.1308	0.0036
94.88	0.454	0.0829	0.0550	0.298	0.1286	0.0109
96.14	0.376	0.1100	0.0286	0.372	0.1125	0.0262
97.40	0.303	0.1270	0.0120	0.450	0.0855	0.0509
98.66	0.236	0.1297	0.0040	0.530	0.0554	0.0820
99.92	0.179	0.1198	0.0010	0.609	0.0299	0.1120
101.18	0.131	0.1012	0.0002	0.684	0.0131	0.1326
102.44	0.093	0.0790		0.751	0.0046	0.1387
103.70	0.064	0.0579		0.811	0.0012	0.1302
104.96	0.042	0.0402		0.860	0.0002	0.1117
106.22	0.027	0.0263		0.900	0.0001	0.0887
107.48	0.018	0.0165		0.931		0.0656
108.74	0.010	0.0099		0.954		0.0459
Σ		0.9999	1.0000		1.0000	1.0001
R		0.0000	0.0000		0.0000	0.0000

Table XXXIV

Series IV

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.922	0.0002	0.0780	0.018	0.0183	
86.48	0.874	0.0014	0.1149	0.034	0.0330	
88.16	0.812	0.0069	0.1514	0.058	0.0552	
89.84	0.733	0.0232	0.1753	0.095	0.0849	0.0004
91.52	0.639	0.0560	0.1736	0.147	0.1186	0.0025
93.20	0.536	0.1011	0.1425	0.215	0.1481	0.0106
94.88	0.430	0.1423	0.0937	0.297	0.1612	0.0318
96.56	0.329	0.1623	0.0474	0.393	0.1496	0.0698
98.24	0.239	0.1552	0.0177	0.496	0.1145	0.1171
99.92	0.165	0.1282	0.0046	0.598	0.0697	0.1561
101.60	0.107	0.0935	0.0008	0.694	0.0326	0.1710
103.28	0.066	0.0615	0.0001	0.779	0.0112	0.1588
104.96	0.038	0.0370		0.848	0.0027	0.1290
106.64	0.021	0.0206		0.902	0.0004	0.0920
108.32	0.011	0.0106		0.939	0.0001	0.0609
Σ		1.0000	1.0000		1.0001	1.0000
R		0.0000	0.0000		0.0000	0.0000

Table XXXV

Series V

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.906	0.0022	0.0944	0.009	0.0091	
86.90	0.838	0.0125	0.1465	0.022	0.0220	
89.00	0.746	0.0437	0.1932	0.049	0.0472	0.0004
91.10	0.631	0.1001	0.2090	0.96	0.0887	0.0041
93.20	0.503	0.1605	0.1775	0.171	0.1421	0.0219
95.30	0.375	0.1912	0.1122	0.274	0.1895	0.0697
97.40	0.259	0.1784	0.0498	0.403	0.2020	0.1424
99.50	0.165	0.1364	0.0145	0.542	0.1624	0.2013
101.60	0.097	0.0884	0.0026	0.677	0.0928	0.2102
103.70	0.052	0.0501	0.0003	0.792	0.0351	0.1711
105.80	0.025	0.0251		0.878	0.0080	0.1143
107.90	0.011	0.0113		0.935	0.0011	0.0646
Σ		0.9999	1.0000		1.0000	1.0000
R		0.0002	0.0000		0.0001	0.0000

Table XXXVI

Series VI

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.890	0.0069	0.1101	0.025	0.0250	
87.32	0.806	0.0321	0.1725	0.055	0.0540	0.0008
89.84	0.692	0.0895	0.2208	0.109	0.1004	0.0069
92.36	0.556	0.1621	0.2204	0.193	0.1580	0.0326
94.88	0.413	0.2047	0.1622	0.307	0.2033	0.0914
97.40	0.280	0.1929	0.0821	0.444	0.2038	0.1652
99.92	0.173	0.1440	0.0264	0.588	0.1502	0.2083
102.44	0.096	0.0884	0.0050	0.721	0.0759	0.1956
104.96	0.048	0.0462	0.0005	0.829	0.0244	0.1446
107.48	0.021	0.0211		0.906	0.0046	0.0881
110.00	0.008	0.0084		0.953	0.0004	0.0458
112.52	0.003	0.0030		0.979		0.0206
Σ		0.9993	1.0000		1.0000	0.9999
R		0.0008	0.0000		0.0000	0.0000

Table XXXVII

Series VII

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.954	0.0085	0.0459	0.009	0.0086	
87.74	0.873	0.0615	0.1208	0.029	0.0292	0.0014
90.68	0.726	0.1861	0.2287	0.082	0.0785	0.0159
93.62	0.523	0.2807	0.2886	0.164	0.1451	0.0879
96.56	0.313	0.2451	0.2170	0.343	0.2534	0.2015
99.50	0.151	0.1395	0.0840	0.536	0.2601	0.2656
102.44	0.058	0.0566	0.0141	0.720	0.1622	0.2221
105.38	0.017	0.0171	0.0009	0.860	0.0541	0.1297
108.32	0.004	0.0039		0.942	0.0083	0.0567
111.26	0.001	0.0007		0.981	0.0005	0.0192
Σ		0.9997	1.0000		1.0000	1.0000
R		0.0004	0.0000		0.0000	0.0000

TABLE XXXVIII

Series VIII

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.938	0.0166	0.0621	0.011	0.0114	0.0003
88.16	0.839	0.0923	0.1508	0.041	0.0406	0.0065
91.52	0.672	0.2252	0.2582	0.115	0.1092	0.0517
94.88	0.460	0.2855	0.2856	0.254	0.2130	0.1716
98.24	0.259	0.2169	0.1803	0.450	0.2819	0.2806
101.60	0.117	0.1105	0.0557	0.661	0.2272	0.2623
104.96	0.041	0.0406	0.0070	0.829	0.0968	0.1590
108.32	0.011	0.0112	0.0003	0.932	0.0185	0.0680
Σ		0.9988	1.0000		0.9986	1.0000
R		0.0011	0.0000		0.0014	0.0000

TABLE XXXIX

Series IX

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.952	0.0287	0.0478	0.004	0.0040	0.0006
88.58	0.840	0.1586	0.1521	0.028	0.0281	0.0200
92.36	0.628	0.3179	0.2980	0.121	0.1175	0.[illegible]488
96.14	0.365	0.2912	0.3188	0.334	0.2837	0.3383
99.92	0.155	0.1463	0.1549	0.622	0.3523	0.3091
103.70	0.046	0.0454	0.0271	0.853	0.1829	0.1408
107.48	0.009	0.0092	0.0013	0.963	0.0304	0.0367
111.26	0.001	0.0012		0.994	0.0011	0.0058
Σ		0.9985	1.0000		1.0000	1.0001
R		0.0014	0.0000		0.0001	0.0000

TABLE XL

Series X

Comparison Stimulus	q	Q	Q¹	p	P	P¹
84.80	0.952	0.0390	0.0483	0.014	0.0140	0.0023
89.00	0.828	0.1978	0.1634	0.061	0.0598	0.0366
93.20	0.592	0.3468	0.3215	0.183	0.1699	0.1733
97.40	0.315	0.2697	0.3196	0.400	0.3022	0.3188
101.60	0.116	0.1123	0.1301	0.653	0.2967	0.2817
105.80	0.028	0.0280	0.0166	0.851	0.1340	0.1418
110.00	0.004	0.0044	0.0005	0.954	0.0224	0.0455
Σ		0.9980	1.0000		0.9990	1.0000
R		0.0020	0.0000		0.0010	0.0000

TABLE XLI

series there are only two remainders and these are only 0.0001. In the later and shorter series the remainders are very much larger and occur much more frequently. It will also be noticed that there are no remainders for either subject for the just imperceptible differences both positive and negative. This of course indicates that our series were in every case extended enough so that these imperceptible differences would always fall upon one of the stimuli of our series, within the limit of 0.0001. It would also seem that the remainders for the just perceptible negative difference are on the whole greater than those of the just perceptible positive difference.

Series	Subject I: Just perceptible negative difference	Subject I: Just perceptible positive difference	Subject II: Just perceptible negative difference	Subject II: Just perceptible positive difference
I.......	0.0000	0.0000	0.0001	0.0000
IV.......	0.0000	0.0001	0.0000	0.0000
V.......	0.0000	0.0001	0.0002	0.0001
VI.......	0.0002	0.0000	0.0008	0.0000
VII.......	0.0006	0.0000	0.0004	0.0000
VIII.......	0.0006	0.0014	0.0011	0.0014
IX.......	0.0006	0.0001	0.0014	0.0001
X.......	0.0031	0.0001	0.0020	0.0010

When we have once obtained the probabilities for the different comparison stimuli with which these four differences may occur, we may derive the most probable value of each of the just perceptible and just imperceptible differences very easily. Multiplying each value of the comparison stimulus by its probability and adding these products we obtain $[xQ]$, $[xQ_1]$, $[xP]$, and $[xP_1]$. These are respectively the most probable values of the just perceptible negative difference, the just imperceptible negative difference, the just perceptible positive difference and the just imperceptible positive difference, calculated on the basis of our experiments by the method of constant stimuli. Tables XLII and XLIII give these values of the four differences for subjects I and II. Opposite the numbers for the series, which are found in the first columns, we have in the 3d, 5th, 7th, and 9th columns the calculated values for these differences.

The observed values of these fundamental differences are found in the other columns of these tables. The observed and calculated values for each series and for every difference, are found directly next to one another so that they may be more easily compared. These observed values are calculated very readily and in the manner explained in our description of the experiment, given above. The values for the just perceptible and imperceptible negative and positive differences are ascertained for each group in every series, by picking out each value according to the definitions of these quantities. There were ten groups in each series, five ascending and five descending, so that we will obtain in each case, ten values for each difference. These ten values are averaged and the result is given as the value of the difference. The ease with which the differences are calculated after we have our experimental data is one of the great advantages of the method of just perceptible differences over the other methods of psychophysical measurement. In this method, the measure of sensitivity is obtained by the simple solving of four averages, which can be performed in a very short space of time. With the method of constant stimuli, on the other hand, a long and complicated series of calculations must be entered into before these same values are obtained.

We now calculate the measure of accuracy that is shown by our observed results of the method of just perceptible differences by means of the probable error. If our threshold is found to be 95 grams and the probable error $\pm$ 0.50 grams, it indicates that this threshold will fall within the interval of 94.50 grams and 95.50 grams with the probability of 0.50. This measure of precision is

Subject I

Series	Just perceptible negative difference		Just imperceptible negative difference		Just perceptible positive difference		Just imperceptible positive difference	
	Observed	Calculated	Observed	Calculated	Observed	Calculated	Observed	Calculated
I.........	99.29	99.74	94.33	92.18	96.31	95.56	100.84	102.58
II.........	99.92	101.02	90.60	89.65	95.38	94.18	103.78	104.35
III.........	97.27	99.31	90.47	90.38	94.63	95.52	101.94	103.38
IV.........	101.10	98.32	93.70	91.66	98.41	96.27	104.12	99.76
V.........	95.51	96.78	91.73	91.96	96.14	96.79	99.71	101.46
VI.........	93.62	96.69	92.41	92.49	96.90	98.01	101.18	101.75
VII.........	97.15	95.68	96.27	92.80	102.44	97.84	102.73	100.30
VIII.........	95.55	95.70	94.21	93.97	100.93	98.73	99.58	99.92
IX.........	96.14	95.15	96.14	94.83	101.81	99.52	101.81	100.03
X.........	93.62	93.98	96.14	94.22	102.44	99.66	99.92	98.20

Table XLII

Subject II

Series	Just perceptible negative difference		Just imperceptible negative difference		Just perceptible positive difference		Just imperceptible positive difference	
	Observed	Calculated	Observed	Calculated	Observed	Calculated	Observed	Calculated
I.........	97.82	98.41	93.66	91.23	94.96	94.77	99.54	101.91
II.........	99.08	100.56	89.76	89.23	94.46	91.76	103.03	104.11
III.........	98.66	99.01	90.60	90.10	93.75	93.43	102.31	102.53
IV.........	98.07	97.70	91.69	90.57	96.06	94.30	100.42	101.84
V.........	96.35	96.47	91.73	90.80	95.09	96.14	99.92	101.15
VI.........	97.65	96.50	94.12	91.18	99.42	93.56	100.68	101.32
VII.........	95.09	94.95	95.97	93.10	98.32	97.61	101.26	100.38
VIII.........	95.55	95.37	96.22	93.43	99.92	97.83	100.26	99.88
IX.........	93.49	94.43	93.12	94.13	100.30	98.21	99.54	98.16
X.........	96.14	94.38	96.98	94.76	99.50	98.66	98.24	99.28

Table XLIII

calculated only for the thresholds and is obtained by the formula

$$\text{P. E.} — 0.6745 \sqrt{\frac{\Sigma v^2}{n\,(n-1)}}.$$

In this equation Σv^2 is the sum of the variations of the different thresholds from the average, and n is the number of thresholds considered. Table XLIV gives the probable errors for both sub-

Series	Subject I		Subject II	
	Threshold in direction of decrease	Threshold in direction of increase	Threshold in direction of decrease	Threshold in direction of increase
I	±0.50	±0.43	±0.43	±0.44
II	±0.88	±0.82	±0.87	±0.75
III	±0.78	±0.83	±0.79	±0.83
IV	±0.96	±0.72	±0.83	±0.61
V	±0.72	±0.66	±0.65	±0.65
VI	±0.27	±0.64	±0.62	±0.52
VII	±0.53	±0.65	±0.50	±0.63
VIII	±0.52	±0.56	±0.57	±0.56
IX	±0.56	±0.60	±0.62	±0.76
X	±0.55	±0.54	±0.60	±0.72

Table XLIV

jects for the two thresholds of increase and decrease. It will be noticed that these values are comparatively constant. Considering the size of the values, the precision of which they are measuring, these quantities are small. In no case is the probable error greater than one per cent of the threshold and very often it is close to one-half a per cent or even less. Thus we conclude that the thresholds in the method of just perceptible differences fall with a high probability within a comparatively limited space.

The sums of the products xP, xP^1, xQ and xQ^1 give the most probable values of the just perceptible and just imperceptible positive and negative differences, but a finite number of observations can not be expected to give exactly these results. The outcome of a series depends on chance influences, which prevent us from obtaining the calculated result exactly, and we must be satisfied with determining the limit inside of which we may expect the results to fall with a given probability. The solution of this problem is given by the theorem of Tchebicheff, which applies to problems of this kind (cfr. Urban, *Statistical Methods,* pp. 65, and *Archiv f. d. ges. Psychologie,* Vol. 15, pp. 302-304). We give the formulae for the just perceptible and the just imperceptible positive difference only, because their form for the two other differences is perfectly obvious. Suppose we make n determinations of each one of these quantities with a certain series of comparison stimuli. We then may expect with a probability exceeding $1 - \frac{t^2}{n}$ that the arithmetical mean of all the observations of the just perceptible difference will be within the limits

$$\Sigma x P \pm \frac{1}{t}\sqrt{\Sigma x^2 P - (\Sigma x P)^2}$$

and that of the just imperceptible positive difference between the limits

$$\Sigma x P^1 \pm \frac{1}{t}\sqrt{\Sigma x^2 P^1 - (\Sigma x P^1)^2}\ .$$

Applying the same theorem to the determination of the upper limit of the interval of uncertainty we find that there exists a probability exceeding $1 - \frac{t^2}{n}$ that the actually observed result will be found between the limits

$$\Sigma x \frac{P+P^1}{2} \pm \frac{1}{t}\sqrt{\Sigma x^2 \frac{P+P^1}{2} - \frac{1}{2}(\Sigma x P)^2 - \frac{1}{2}(\Sigma x P^1)^2}\ .^*$$

The difference $\Sigma x^2P - (\Sigma xP)^2$ plays a part similar in the theorem of Tchebicheff to that of the product 2spq in the theorem of Bernoulli, since the square roots of both these terms determine the size of the interval inside of which the result may be expected with a given probability. The actual determination of these intervals requires the square root of these differences, but this problem is of less interest than the question as to the comparative size of these intervals in series of different length. For this purpose it is sufficient to give these differences as it is done in Table XLV for Subject I and in Table XLVI for Subject II. These tables also contain the values of the sums of the terms xxP which is needed for the calculation of the differences in question. These quantities also enable us to determine the limits for the result of a series of determinations of the upper

* The theorem of Tchebicheff enables us merely to determine the limits inside of which we may expect the result of an actual observation to fall with a given probability. Wirth [*Zur erkenntnisstheoretischen und mathematischen Begründung der Massmethoden für die Unterschiedsschwelle, Archiv f. d. ges. Psychologie,* 1911, Vol. 20, p. 84] seems to believe that this theorem might enable us to find relations between different values of the psychometric functions, but there is no possibility of doing so.

Series	ΣxxQ	d_1	ΣxxQ_1	d_2	ΣxxP	d_3	ΣxxP_1	d_4
I.......	9956.04	7.99	8505.99	8.84	9138.58	6.87	10530.34	9.74
II.......	10219.03	13.99	8044.26	7.14	8880.87	10.97	10895.99	7.07
III.......	9875.88	13.40	8179.67	11.13	9138.83	14.76	10697.37	9.95
IV.......	9683.12	14.29	8388.13	12.24	9283.47	15.86	10673.71	11.08
V.......	9381.94	15.58	8469.76	13.12	9383.45	12.74	10306.87	12.74
VI.......	9368.32	19.37	8571.18	16.78	9624.59	18.63	10361.20	19.21
VII.......	9168.66	18.80	8629.89	18.05	9591.46	16.84	9961.88	18.79
VIII.......	9172.02	13.53	8850.82	20.46	9754.65	7.04	10002.67	18.66
IX.......	9069.85	16.32	9010.68	17.95	9924.26	20.03	10027.36	21.36
X.......	8854.31	22.07	8898.74	21.33	9949.78	17.66	9646.89	3.65

TABLE XLV

Series	ΣxxQ	d_1	ΣxxQ_1	d_2	ΣxxP	d_3	ΣxxP_1	d_4
I.......	9689.23	4.70	8331.91		8987.28	5.93	10396.16	10.51
II.......	10124.14	11.83	7969.86	9.00	8424.96	5.06	10847.20	8.31
III.......	9816.76	13.78	8127.18	7.86	8743.51	14.34	10525.39	12.98
IV.......	9559.64	16.30	8214.14	11.22	8910.34	17.85	10383.03	11.64
V.......	9324.30	17.84	8257.00	12.36	9258.82	14.97	10245.52	14.20
VI.......	9330.03	17.78	8329.84	16.04	8774.20	20.72	10287.80	20.01
VII.......	9029.72	14.22	8672.56	13.33	9548.02	20.31	10094.21	18.06
VIII.......	9105.68	10.24	8748.01	18.85	9579.07	8.36	9996.06	19.04
IX.......	8924.32	7.29	8878.78	18.32	9656.84	11.64	9653.37	17.99
X.......	8924.08	16.49	9001.06	21.60	9750.96	17.16	9881.80	25.26

TABLE XLVI

limit of the interval of uncertainty. In our experiments we have in all the cases n = 5, so that the radicals for the limits of the interval of uncertainty can be found by averaging the corresponding values in the tables.

Table XLVII gives these values for both subjects. The greater

Series	Subject I		Subject II	
	d^1	d^{11}	d^1	d^{11}
I.......	8.42	8.30	6.85	8.22
II.......	10.56	9.02	9.84	6.68
III.......	12.26	12.36	11.48	13.66
IV.......	13,26	13.47	13.76	14.74
V.......	14.30	12.74	15.10	14.58
VI.......	18.08	18.92	16.91	20.36
VII.......	18.42	17.82	13.78	19.18
VIII.......	17.00	12.85	14.54	13.70
IX.......	17.14	20.70	12.80	14.82
X.......	21.70	10.66	19.04	21.21

Table XLVII

the value, the greater will be the interval within which we may expect our determination of the threshold to fall with a given probability and therefore the less accurate that determination will be. These quantities definitely tend to increase in size as the series become shorter, and therefore, the determinations of our thresholds by short series of experiments are not so accurate as those obtained by more extended series. Furthermore, it would definitely appear that the determinations of our thresholds by the graded series I are more accurate than any of the others.

We are now in the possession of all the data necessary for the comparison of the two methods under discussion. The thresholds are the important values in these methods, since they are the basis for the determination of the sensitivity of the subject. It is by the means of the values of the thresholds, then, that we will compare the methods of just perceptible differences and of constant stimuli. Tables XLVIII and XLIX show the values of the thresholds for both subjects, obtained in the different ways that have been described above. The first columns in these tables give the numbers of the series into which the results of the method of just perceptible differences were divided. The values of the threshold in the direction of decrease are found in the next three columns. The first of these are the values obtained from the re-

Subject I

Series	Threshold in the direction of decrease			Threshold in the direction of increase		
	Method of constant stimuli	Method of Just Perceptible Differences		Method of constant stimuli	Method of Just Perceptible Differences	
		Calculated	Observed		Calculated	Observed
I........	95.02	95.96	96.81	100.10	99.07	98.58
II........	95.02	95.33	95.26	99.82	99.26	99.58
III........	94.52	94.85	93.87	99.77	99.45	98.28
IV........	94.81	94.99	97.40	100.30	99.76	101.26
V........	94.20	94.37	93.62	99.30	99.13	97.92
VI........	94.48	94.59	93.02	99.93	99.88	99.04
VII........	94.06	94.24	96.71	99.09	99.07	102.58
VIII........	94.74	94.83	94.88	99.45	99.33	100.26
IX........	94.75	94.99	96.14	99.82	99.78	101.81
X........	93.89	94.10	94.88	98.93	98.93	101.18

Table XLVIII

Subject II

Series	Threshold in the direction of decrease			Threshold in the direction of increase		
	Method of constant stimuli	Method of Just Perceptible Differences		Method of constant stimuli	Method of Just Perceptible Differences	
		Calculated	Observed		Calculated	Observed
I........	93.26	94.81	95.74	98.94	98.34	97.25
II........	94.44	94.89	94.42	98.96	97.94	98.74
III........	94.16	94.55	94.63	98.19	97.98	98.03
IV........	93.76	94.13	94.88	98.32	98.07	98.24
V........	93.24	93.63	94.04	98.86	98.64	97.50
VI........	93.34	93.84	95.88	98.37	97.44	100.05
VII........	93.93	94.02	95.53	98.96	98.99	99.79
VIII........	94.27	94.40	95.88	99.00	98.85	100.09
IX........	94.20	94.28	93.30	98.34	98.18	99.92
X........	94.57	94.57	96.56	99.04	98.97	98.87

Table XLIX

sults of the method of constant stimuli, that were taken simultaneously with the different series of the method of just perceptible differences. In the next column are the calculated thresholds of the method of just perceptible differences, and in the third column are the observed values for the same method. The second halves of the tables show the same arrangement of values for the threshold in the direction of increase.

An examination of these tables shows unsystematic variations between the different results that are to be expected. That is, within a given set, some individual results are greater and some are smaller than the average, but these variations occur in a haphazard manner and not in a regular way. The larger the amount of data that goes to determine any value or set of values, the smaller these chance variations should become. This is borne out by our results, as the random variations are greatest in the observed results of the method of just perceptible differences, which values are determined by smaller experimental data than are the others. But these variations are comparatively small and seem to point to a high degree of similarity between the two methods.

There is one observation, however, that may be of significance. If we compare the observed thresholds of the method of constant stimuli and the calculated thresholds of the method of just perceptible differences, we find that the variations, in the case of both subjects, are always in one direction. The calculated thresholds in the direction of decrease by the method of just perceptible differences are constantly larger than the corresponding values in the method of constant stimuli; and for the threshold in the direction of increase, they are constantly smaller. Thus the limits of the interval of uncertainty are constantly narrowed in the just perceptible difference method when compared with the same interval of the other method. This is a serious fault as it is just this interval that is the measure of sensitivity of our subject. If the values of one method had been constantly greater than those of the other method for both thresholds, the interval of uncertainty would have remained the same although the point of subjective equality would have changed. It will be noticed that these systematic variations are greatest in the long series of short steps, while in the short series of large steps they become so small that they can be practically disregarded. The variation is greatest in series I, which was performed in an effort to test out a graded approach of the central values of the series. This may be due to the fact that by a carefully graded approach, we emphasize before hand the probability that the differences will fall on the central values. The greater the number of results that go to make up a value, the more nearly should that value coincide with a calculated probability. If this be true, then the values of the series I to IV should be more nearly correct than the series VIII to X. It is a curious fact that the more nearly the experimental arrangement of the method of just perceptible differences approaches that of the method of constant stimuli, the closer do the values under discussion coincide. The experimental arrangement of the method of constant stimuli consists of a small number of pairs of stimuli with comparative large differences of intensity in the steps of the series. This is like the arrangement of series X, and here the values of the thresholds for the two methods practically agree.

Series II has quite a different arrangement; having many steps of a small interval and here the greatest discrepancies are found between the two methods. Thus there seem to be discrepancies that may be due, either to the length of the series, or to the attitude with which the subject approached the two methods. Our form of experimentation tried to eliminate any difference in attitude and was probably almost entirely, but not absolutely, successful. If we had been able to devise a means by which the attitude of the subject was identical for both methods, it seems very probable that our results would have showed a still closer agreement.

In practically all of the studies by these methods, it has been found that a higher sensitivity was obtained by the method of just perceptible differences than by the method of constant stimuli. If the difference in our values under discussion are significant, we could then account for the discrepancies in the results of the former studies, as a fundamental difference in the methods themselves.

But these differences are not large enough to have us disregard the method of just perceptible differences as a practical method of psychophysical measurement. It seems wisest, however, to disregard the form of a graded approach of the central values, as the method in this form seems to show its greatest discrepancies. It would also seem that a short series of large steps is preferable to a long series of small steps. The best form of experimentation with this method would be to take a considerable number of series, each series to consist of not more than ten steps and with a considerable difference of weight between the steps. If such a series is used, the results of this study seem to indicate, that any discrepancies between the results obtained in this way and those of the method of constant stimuli taken under identical conditions would be so small that they could be disregarded.